Zero-free Regions and Concentration Inequalities for Hypergraph Colorings in the Local Lemma Regime

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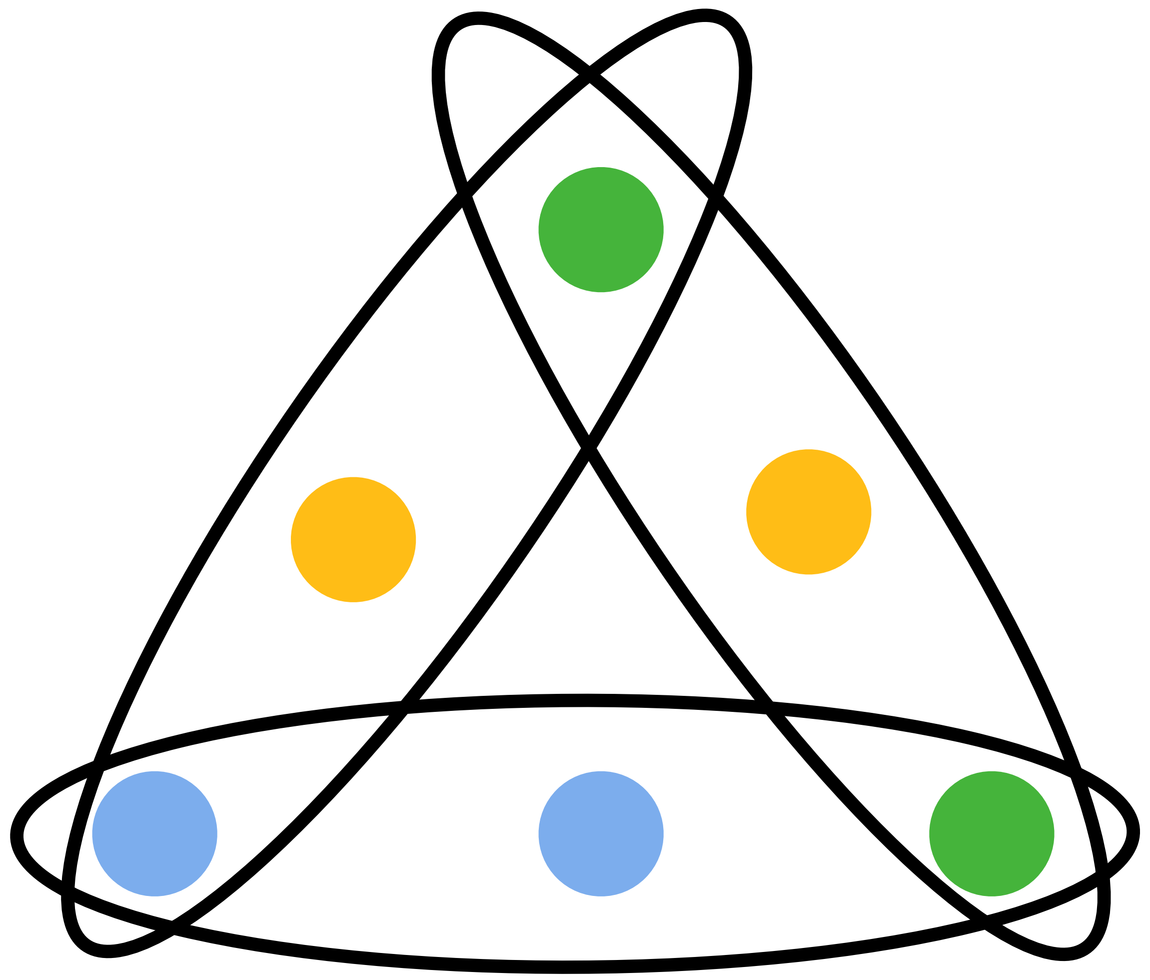
Hypergraph q-colorings

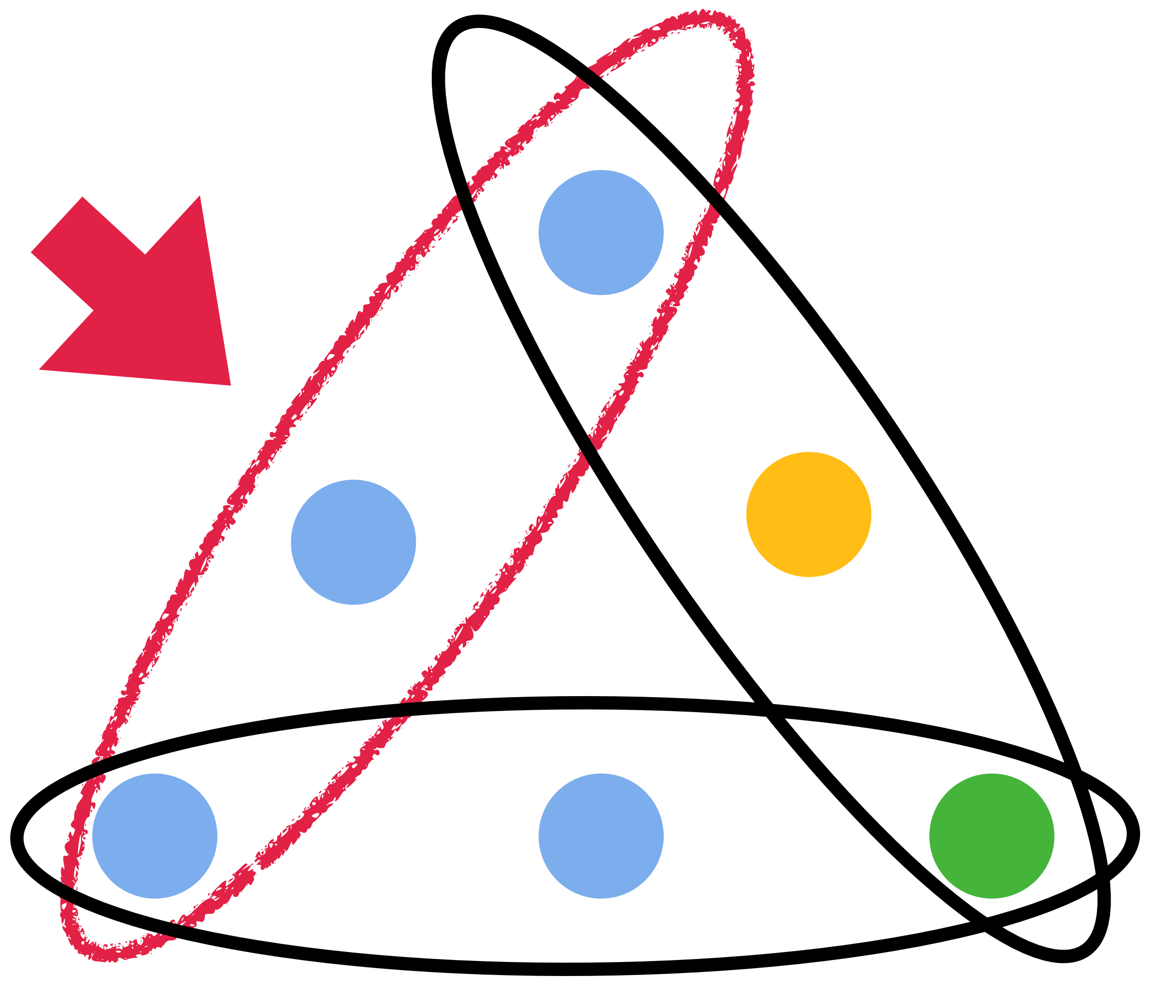
Hypergraph q-coloring:

an integer q, and a k-uniform hypergraph $H = (V, \mathscr{E})$ with the maximum degree Δ.

proper coloring:



improper coloring:



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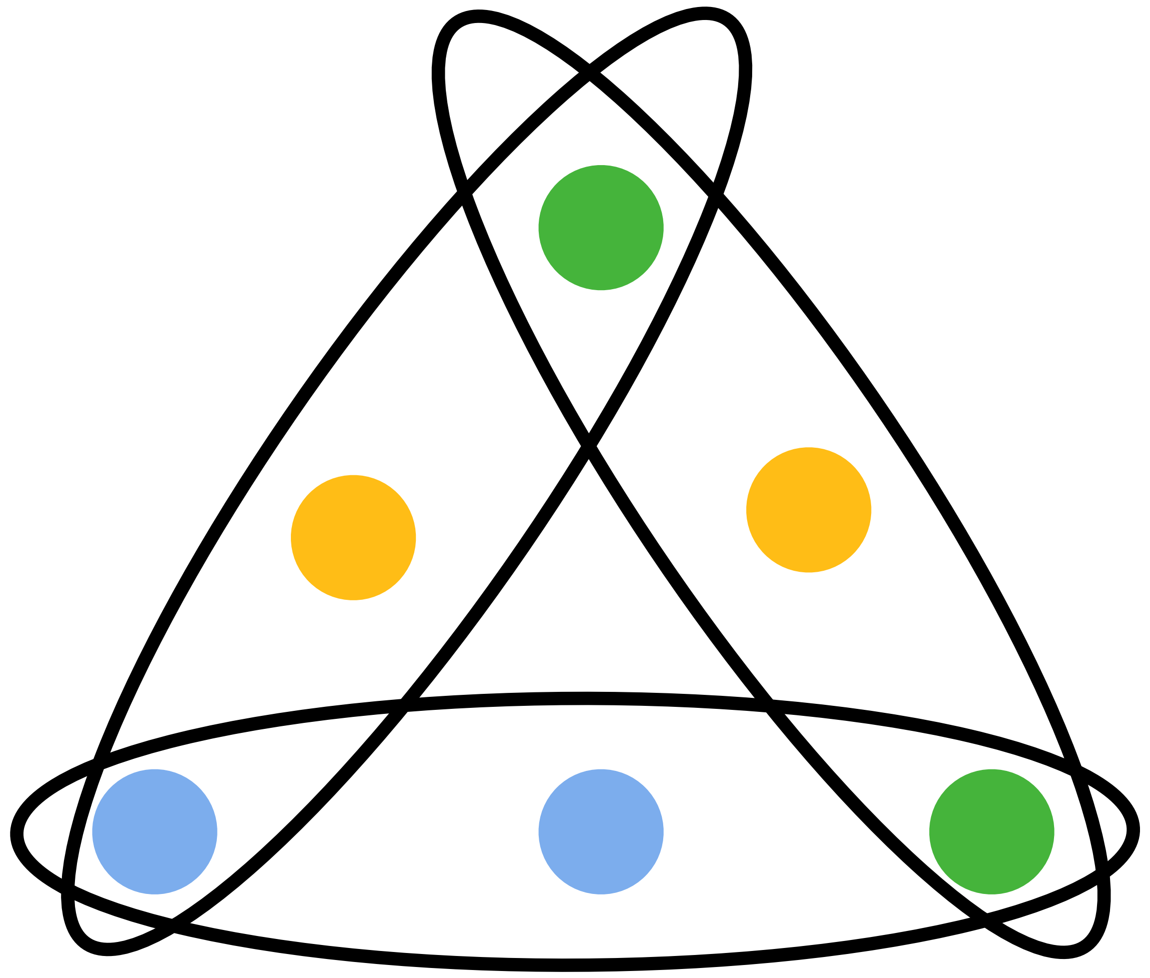

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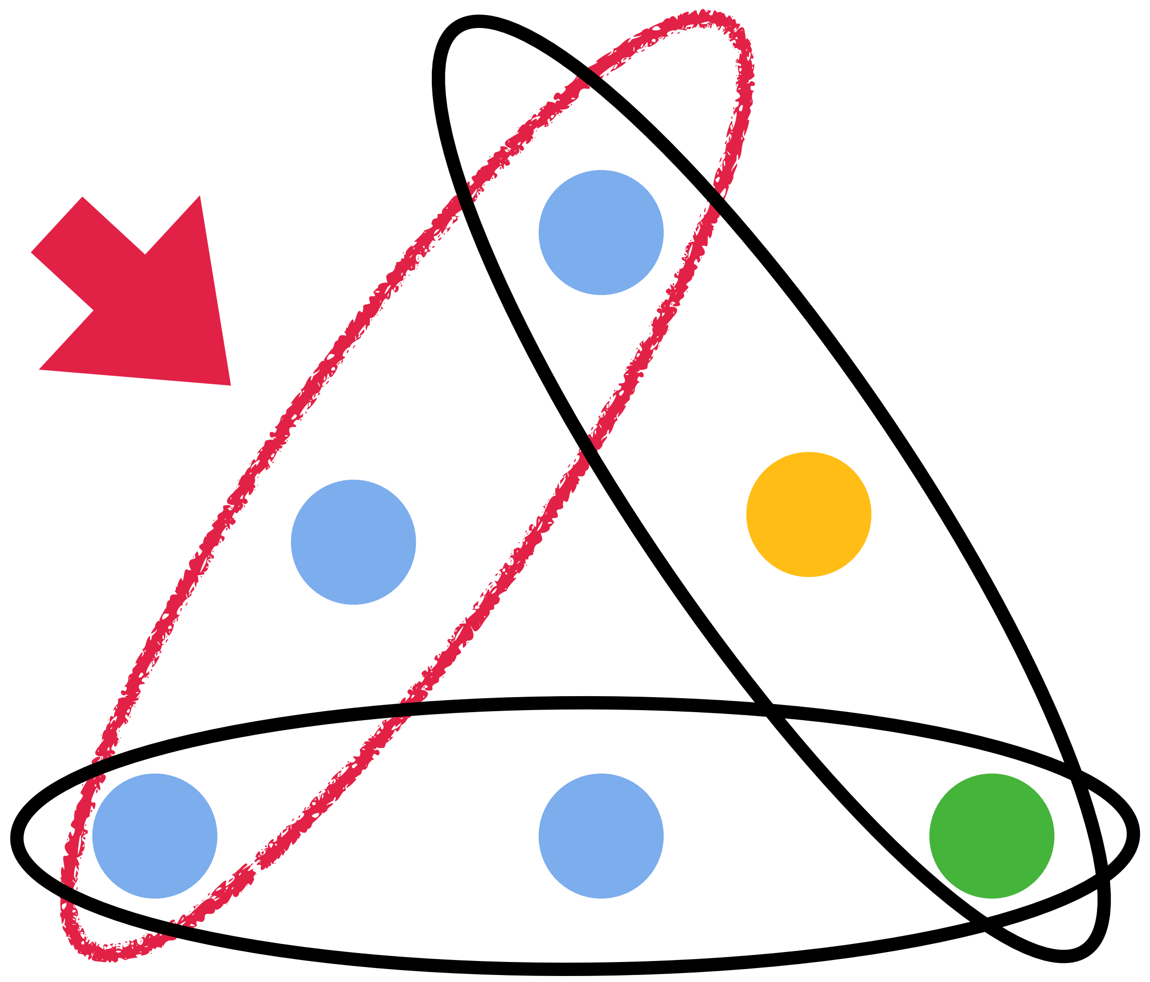


Lovász local lemma (LLL): when $q \geq C\Delta^{\frac{1}{k-1}}$, there exists at least one proper coloring [EL75].

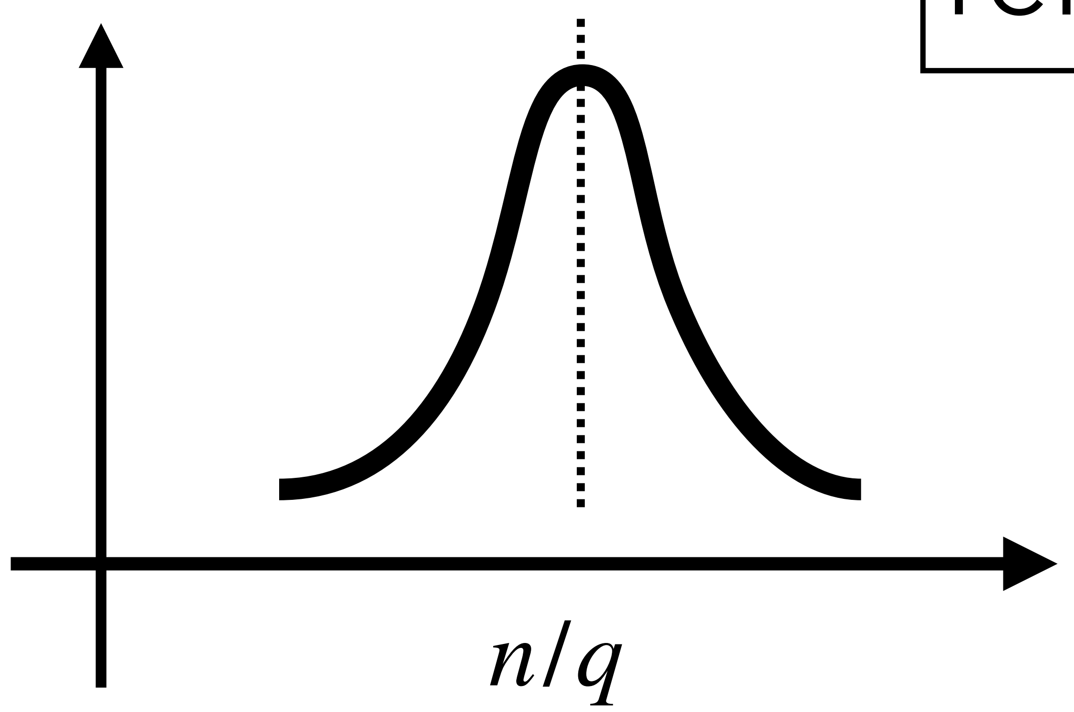
(LLL regime)

Hypergraph q-colorings in LLL regime

We focus on the **concentration phenomenon** in LLL regime**.**

(arbitrary by symmetry)

number of vertices in uniform random proper colorings:



relaxation of independence

Concentration inequalities, central limit theorem (CLT)?

Hypergraph q-colorings in LLL regime

Lovász local lemma (LLL) regime: $p \cdot D^c \lesssim 1$

p: violating probability $\rightarrow$ q^{-k+1}: probability of a hyperedge being monochromatic.

D: dependency degree $\rightarrow$ Δk: dependency degree.

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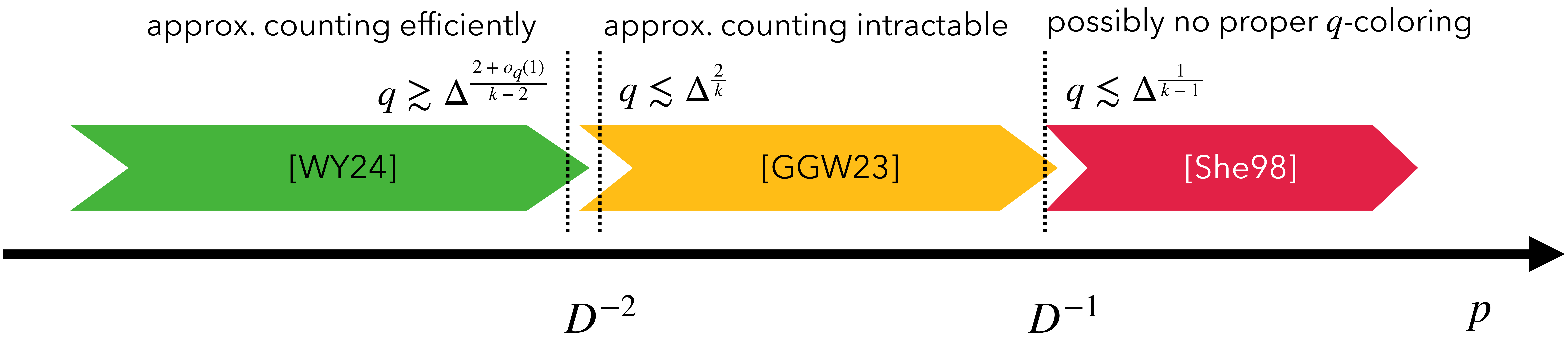


Hypergraph q-colorings in LLL regime

Lovász local le

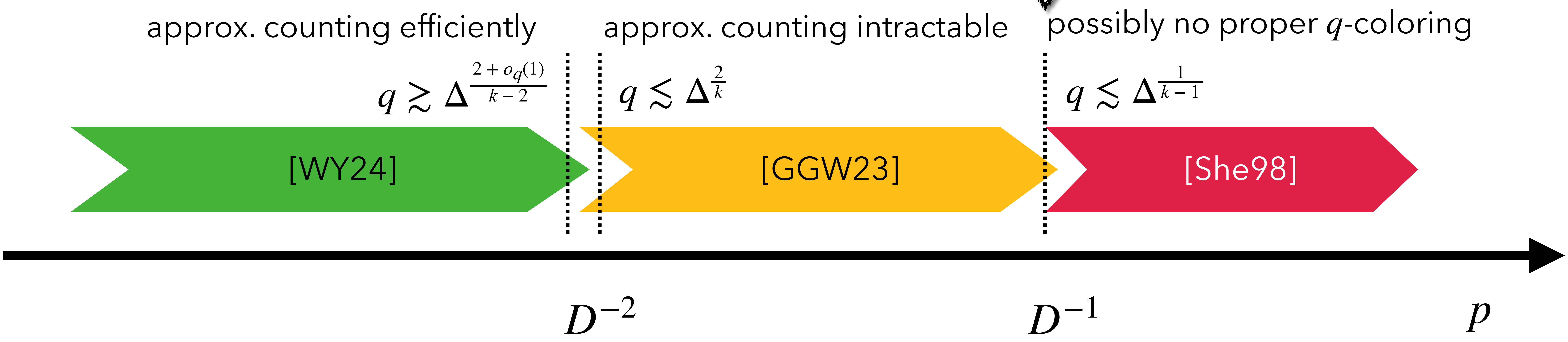
p: violating pro … chromatic.

D: dependenc

Below: LLL guarantees the existence of a proper q-coloring [EL75].

Moser and Tardos give a linear time searching algorithm [MT10].

Above: Shearer shows the tightness [She98].



Hypergraph q-colorings in LLL regime

Lovász local lemm

p: violating probab ng monochromatic.

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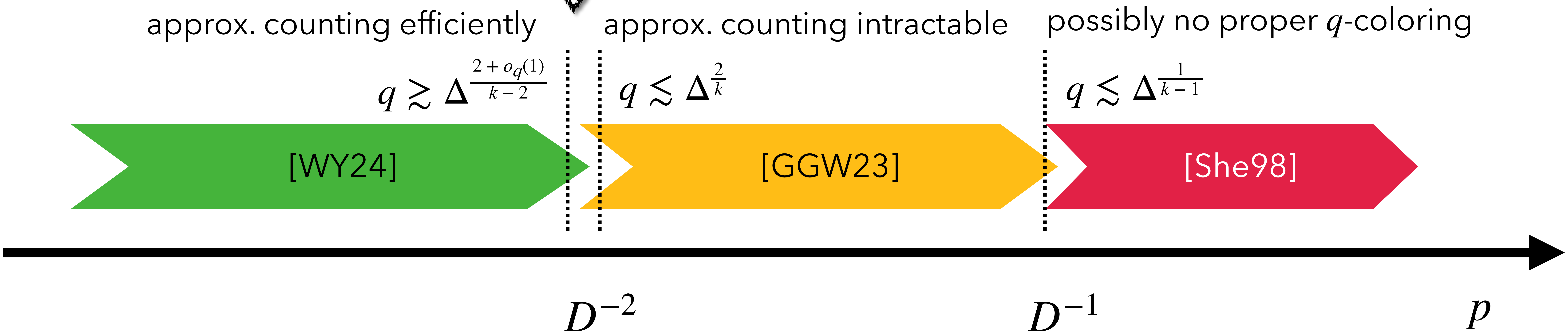
Below: recursive coupling + LP-based FPTAS [WY24].

Above: NP-hard for approx. counting [GGW23].



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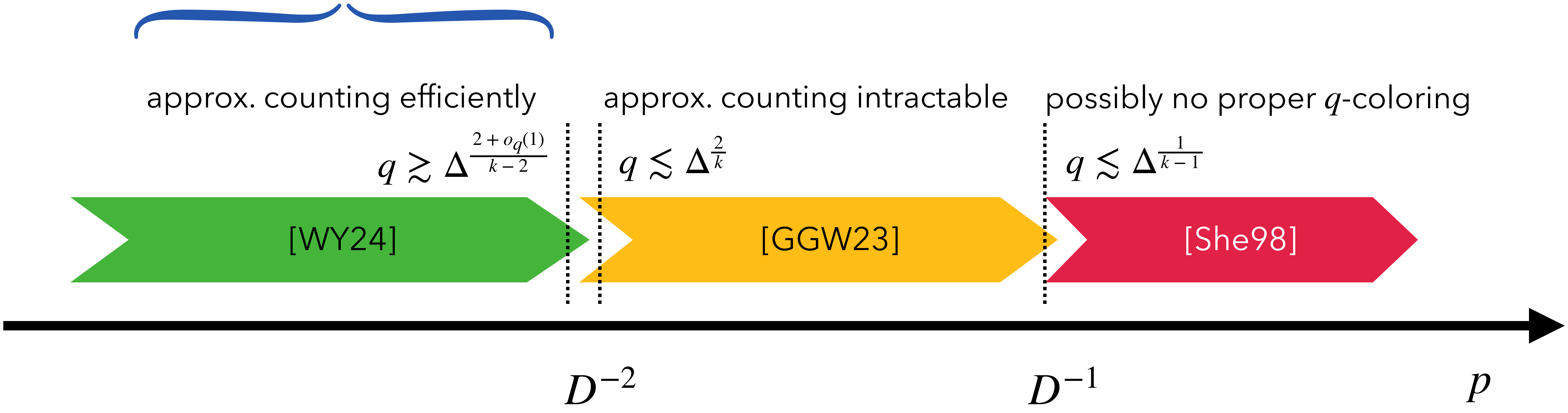
bounded total influence + local uniformity lead to a Chebyshev-type inequality.



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Central limit theorem?

approx. counting efficiently

$q \gtrsim \Delta^{\frac{2+o_q(1)}{k-2}}$

[WY24]

approx. counting intractable

$q \lesssim \Delta^{\frac{2}{k}}$

[GGW23]

possibly no proper q-coloring

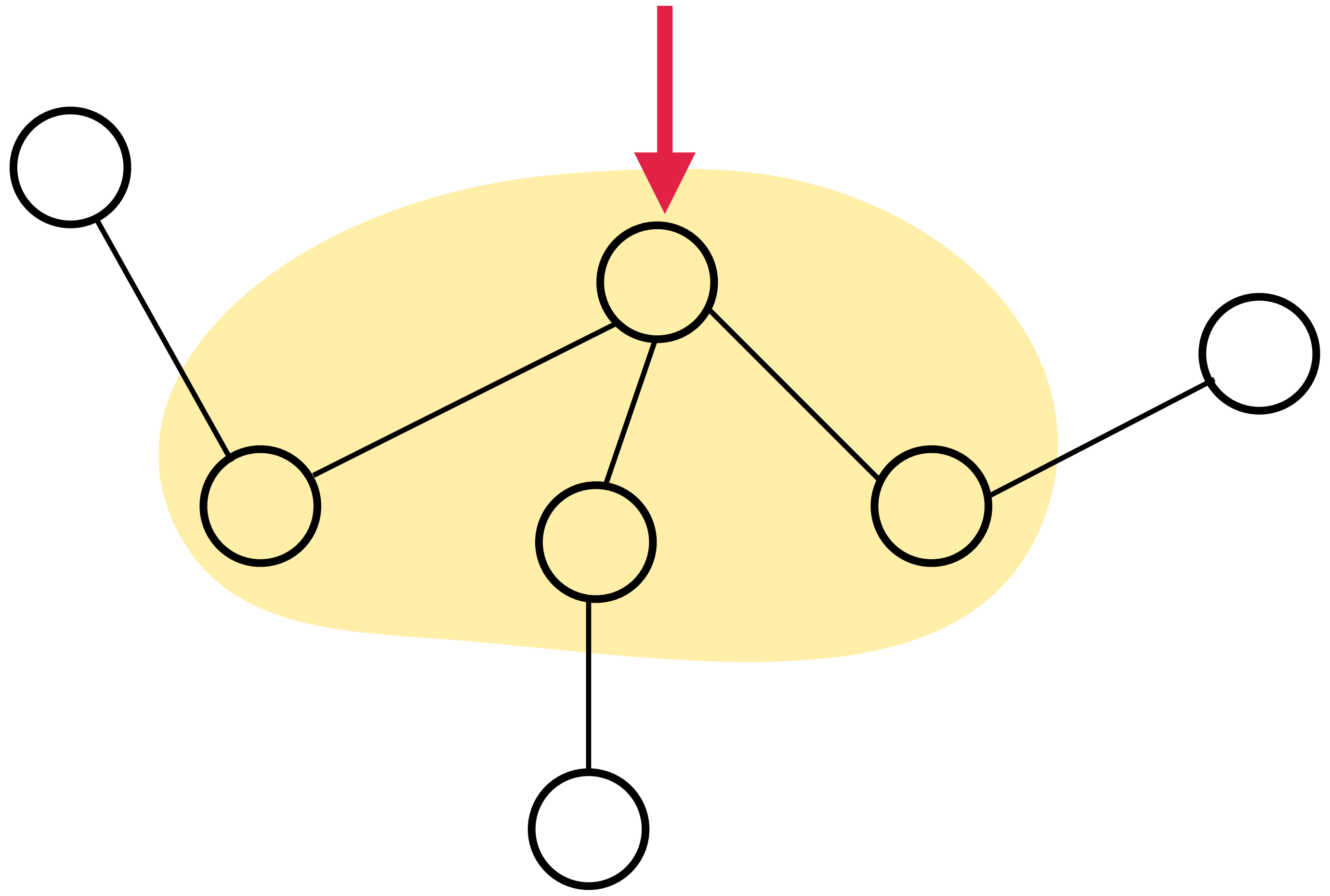
$q \lesssim \Delta^{\frac{1}{k-1}}$

[She98]

D^{-2} $\quad$ D^{-1} $\quad$ p

CLTs beyond product distributions

Stein's method: CLTs under local dependency, e.g. [CS04, CHA08]:



dependency graph

Only yields a CLT about the number of monochromatic hyperedges
under i.i.d. random color assignments.

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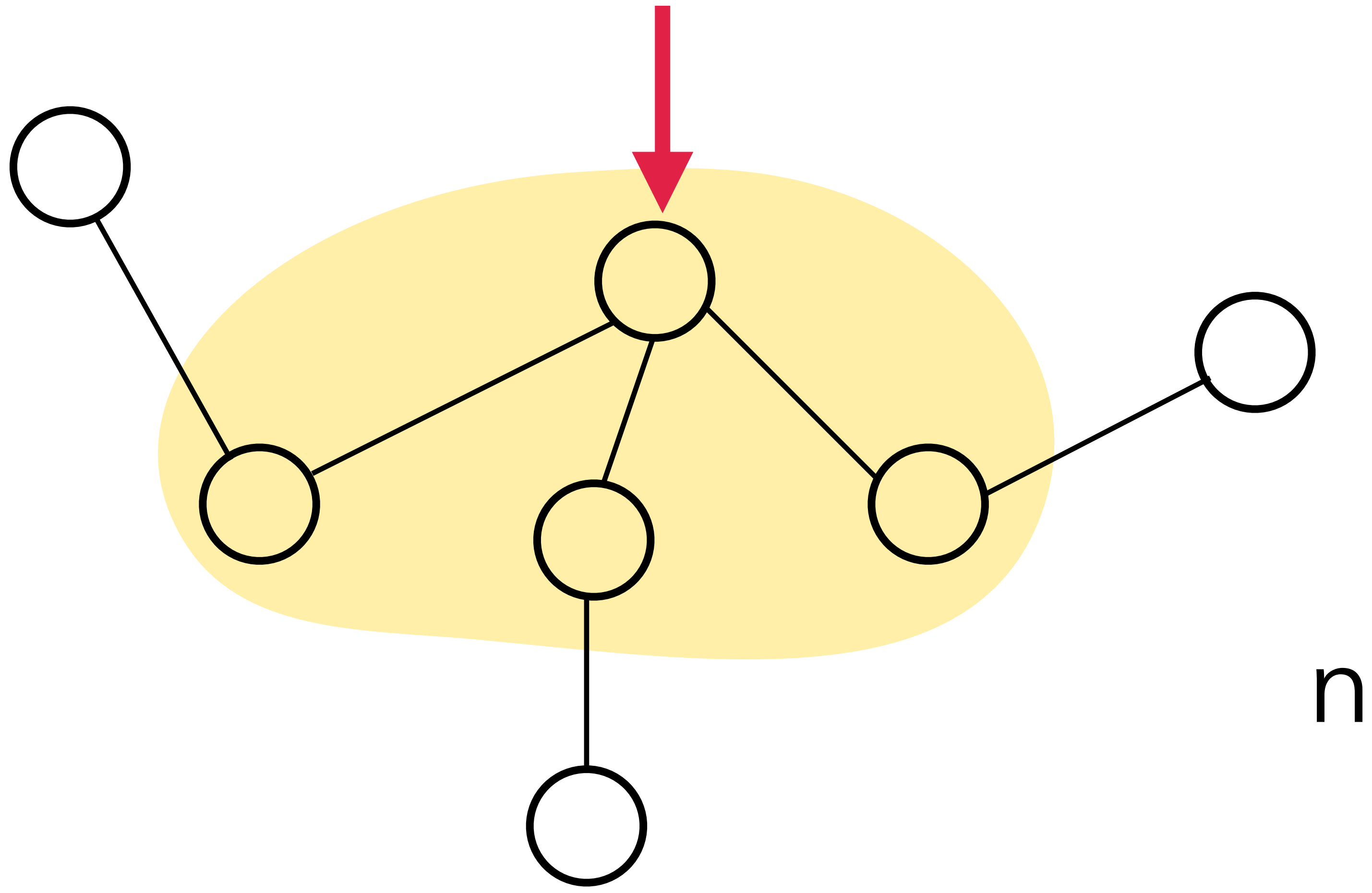


dependency graph

number of ● vertices in uniform random proper colorings

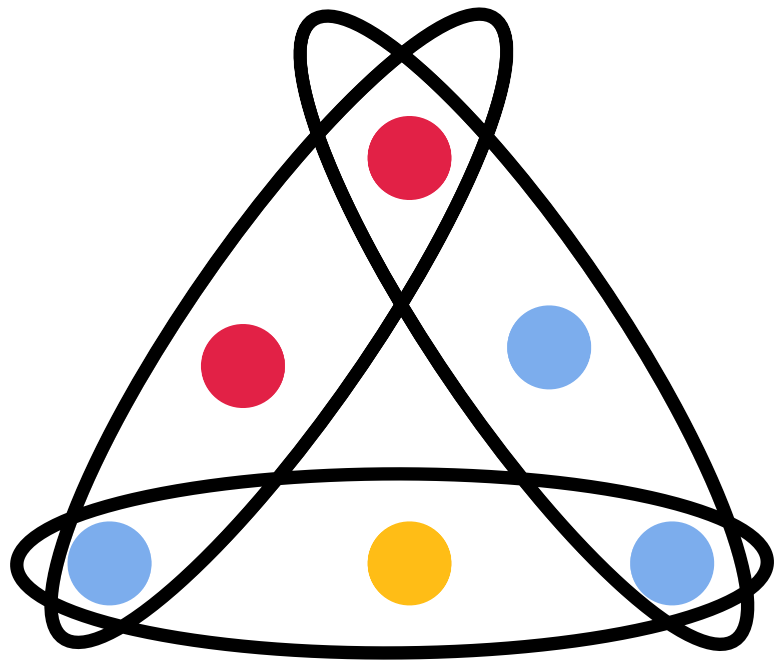
Only yields a CLT about the number of monochromatic hyperedges under i.i.d. random color assignments.

CLTs from Zero-freeness - Lee-Yang zeros

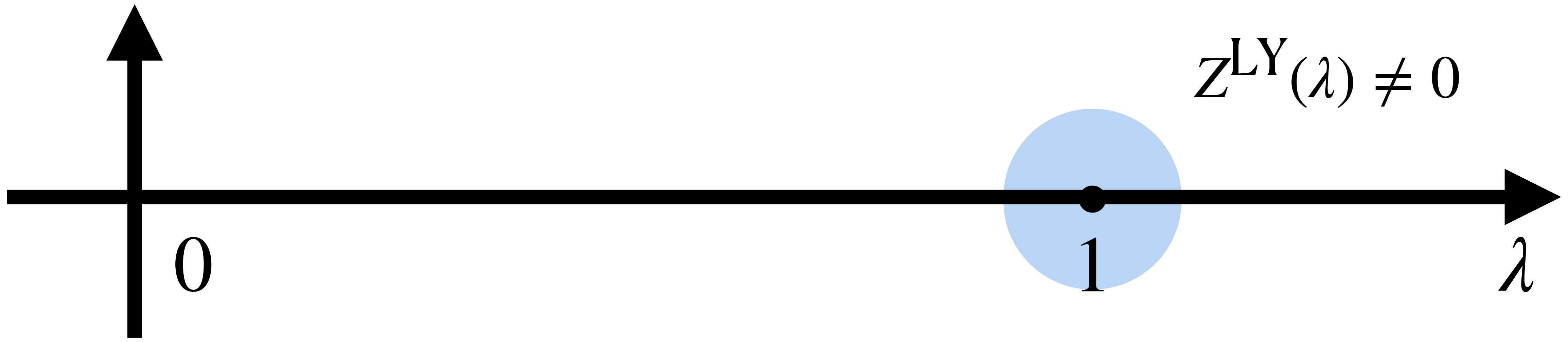
Ω: the set of proper q-colorings.

λ: complex external field on ●.

Partition function w.r.t. Lee-Yang zeros: $Z^{\text{LY}}(\lambda) = \sum_{\sigma \in \Omega} \lambda^{\sharp(\bullet)\text{ in }\sigma}$.



weight: λ^2



[LPRS16, MS24, JPSS22]

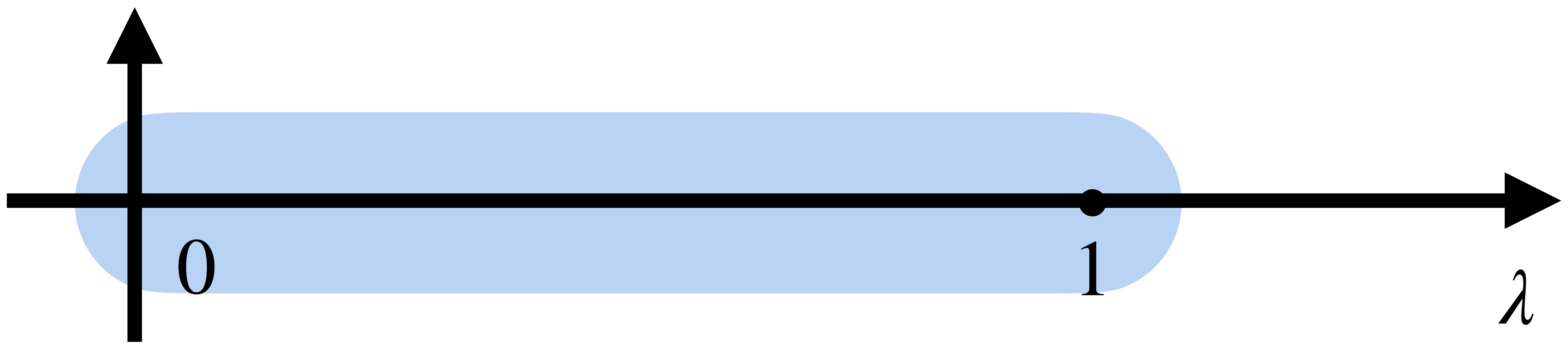
Central limit theorem, local central limit theorem for the number of ● vertices in uniform random proper colorings.

Zero-freeness in LLL regime

Hypergraph independent set on k-uniform hypergraph with maximum degree Δ.

When $\Delta \lesssim 2^{k/2}$:

$(pD^2 \lesssim 1)$



[LWYY25]: convergent complex Markov chain

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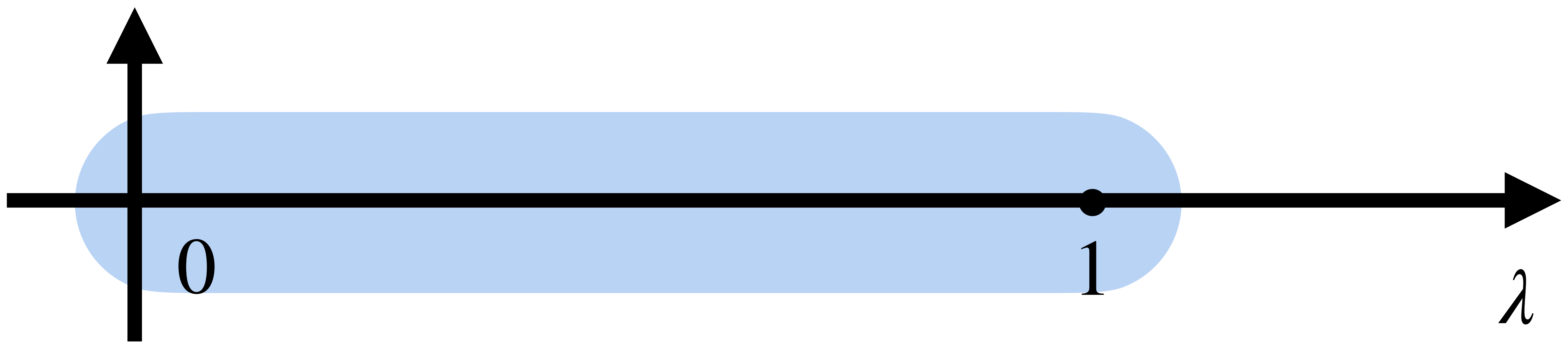

[LWYY25]: convergent complex Markov chain

However, hypergraph colorings (or more general CSPs) have disconnected state space.

For approx. sampling, this is bypassed by the state-compression method [GLLZ18, FHY21].

Our results - Lee-Yang zeros

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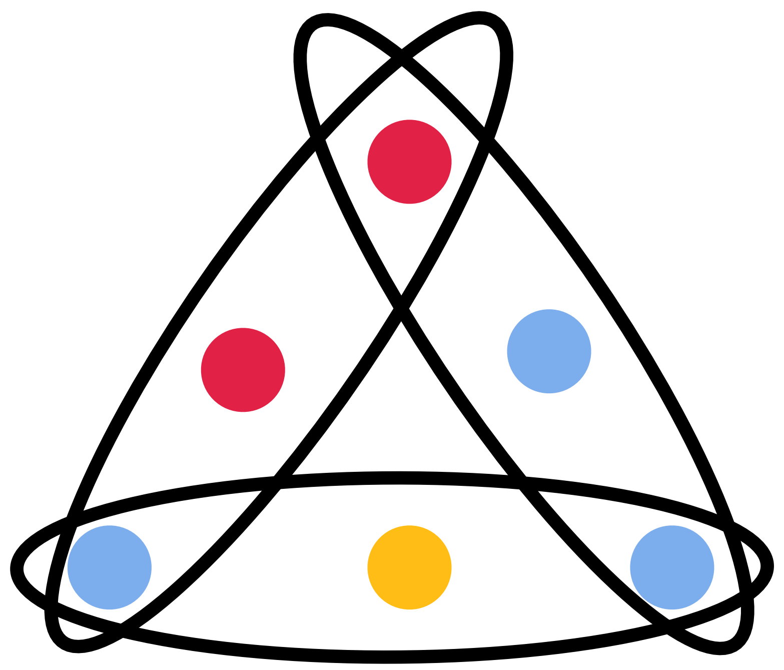
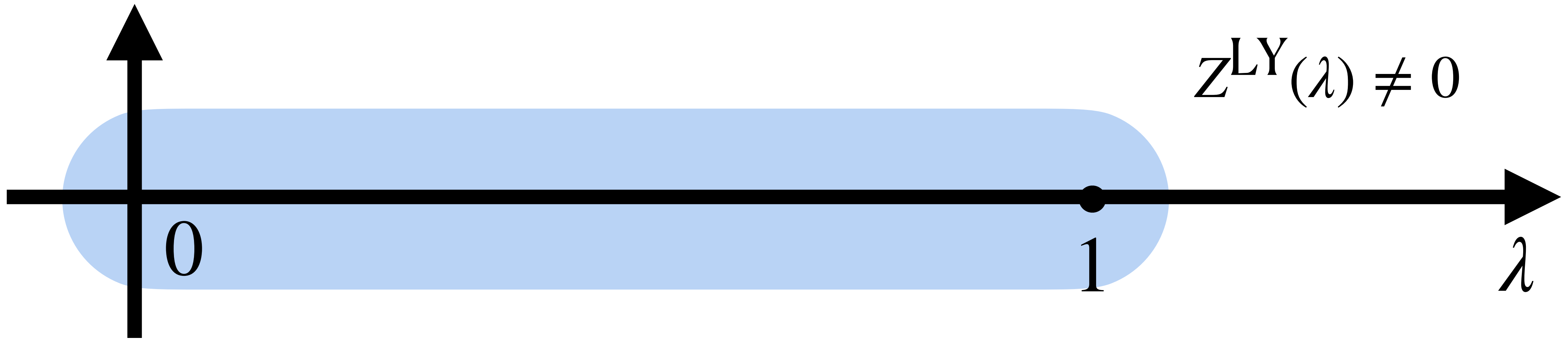


weight: λ^2

When $q \gtrsim \Delta^{\frac{5}{k-2.5}}$:

$(pD^5 \lesssim 1)$



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Central limit theorem, local central limit theorem for the number of ● vertices in uniform random proper colorings.

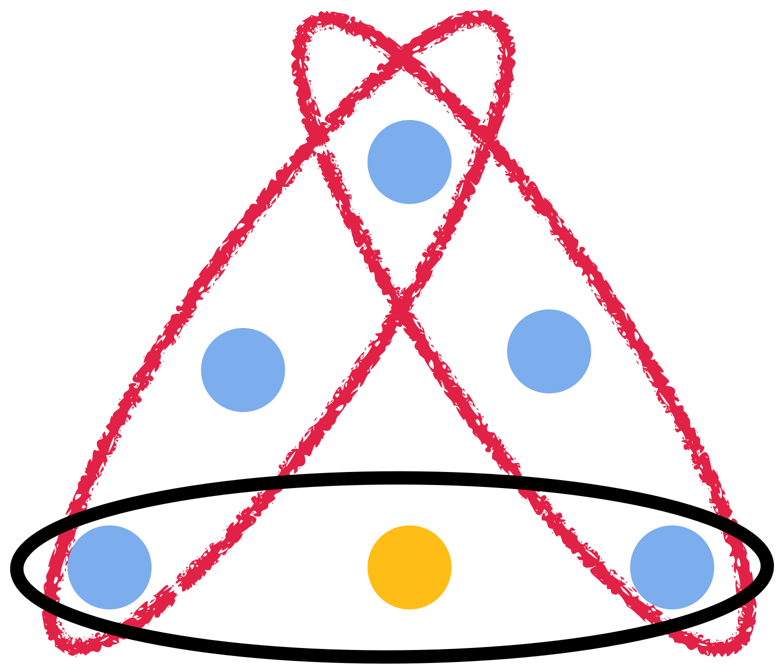
Zero-freeness - Fisher zeros

$$Z^{\mathsf{FS}}(\beta) = \sum_{\sigma \in [q]^V} \beta^{m(\sigma)}$$ partition function w.r.t. Fisher zeros

$m(\sigma)$ = number of monochromatic hyperedges

$Z^{\mathsf{FS}}(0)$ = number of proper colorings (related to approx. counting)

$Z^{\mathsf{FS}}(1) = q^n$ (number of color assignments)



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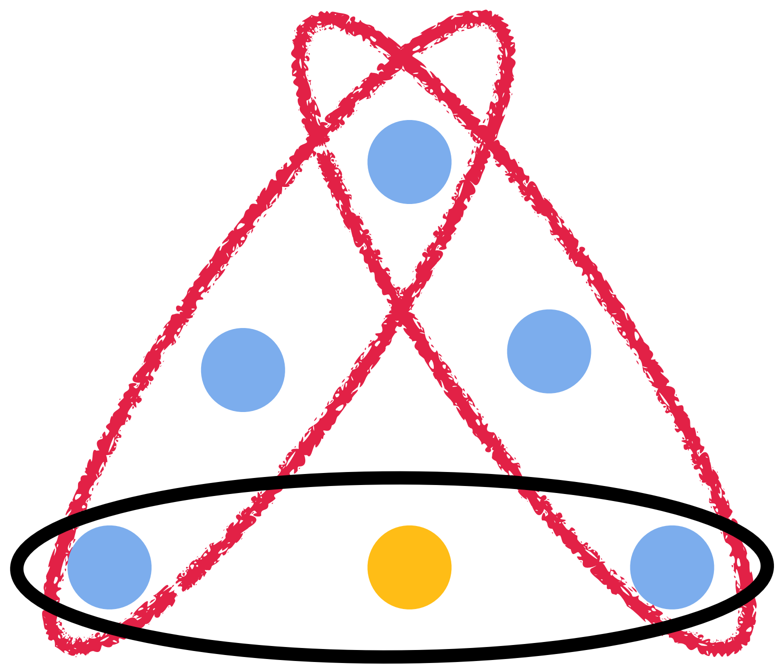

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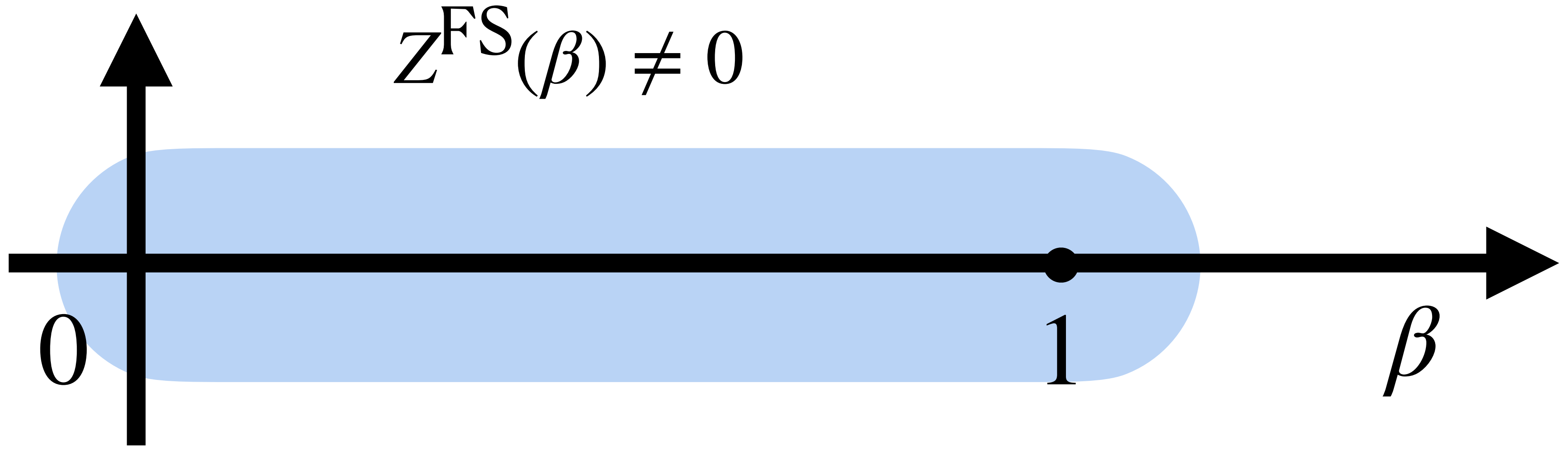
reduction

Fisher zeros $\longrightarrow$ Lee-Yang zeros

When $q \gtrsim \Delta^{\frac{5}{k-10}}$:

$(pD^5 \lesssim 1)$

$Z^{\mathrm{FS}}(\beta) \neq 0$

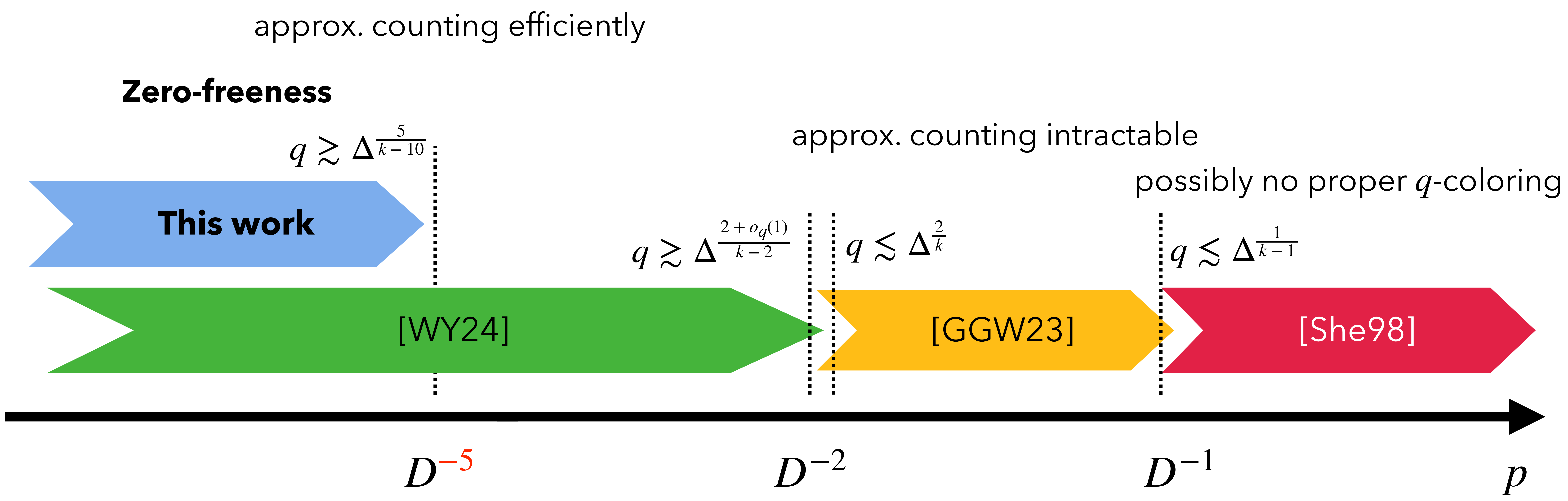


Interpolation

[Bar16, PR17, LSS17]

FPTAS

Our results

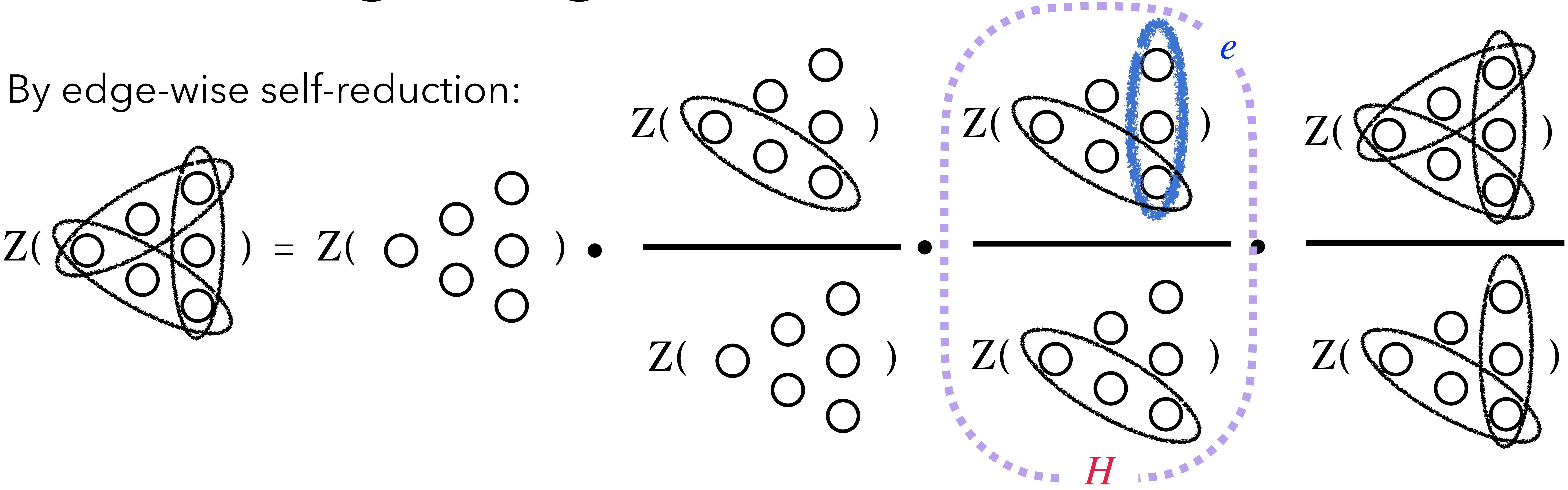


Corollaries: central limit theorem, local central limit theorem and new FPTAS.

Proof overview

Controlling marginal measures

By edge-wise self-reduction:



complex measure: $1 - \mu_H(e \text{ is monochromatic})$

(Note e is not in H)

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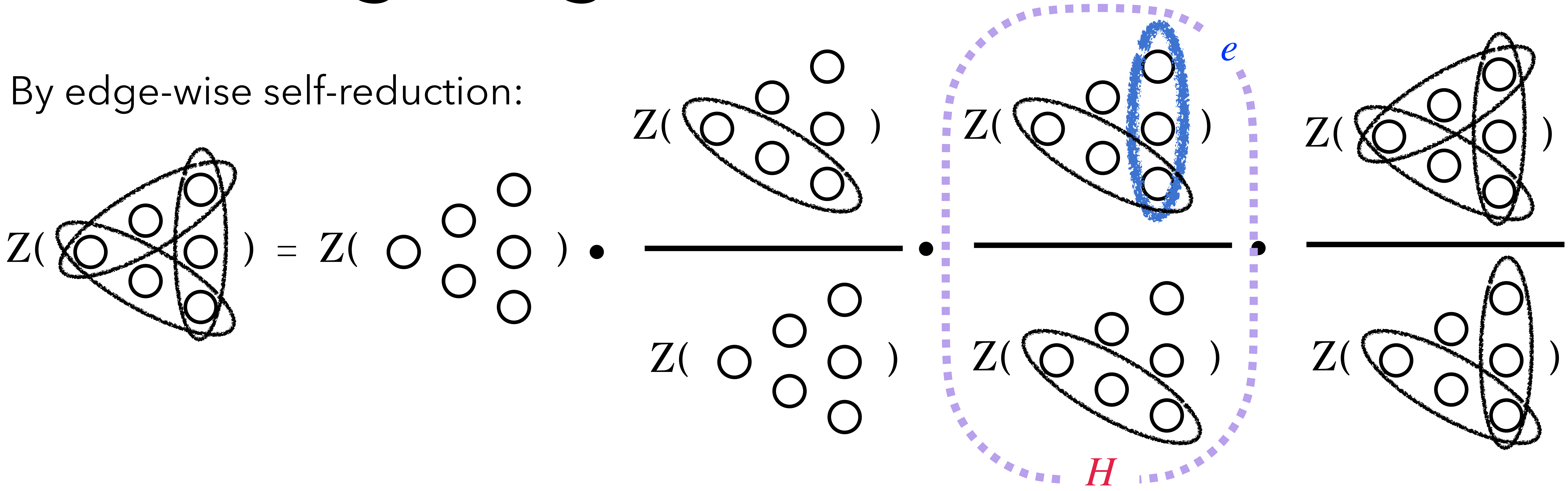


complex measure: $1 - \mu_H(e \text{ is monochromatic})$

(Note e is not in H)

$|\mu_H(e \text{ is monochromatic})| < 1$

(controlling marginal measures) ⟶ zero-freeness

Controlling marginal measures

In [LWYY25], marginal measures are represented through a complex Markov chain.

$$\mu_H(e \text{ is monochromatic}) = \sum \begin{array}{c}\text{percolation analysis on a convergent} \\ \text{complex Markov chain}\end{array}$$

Controlling marginal measures

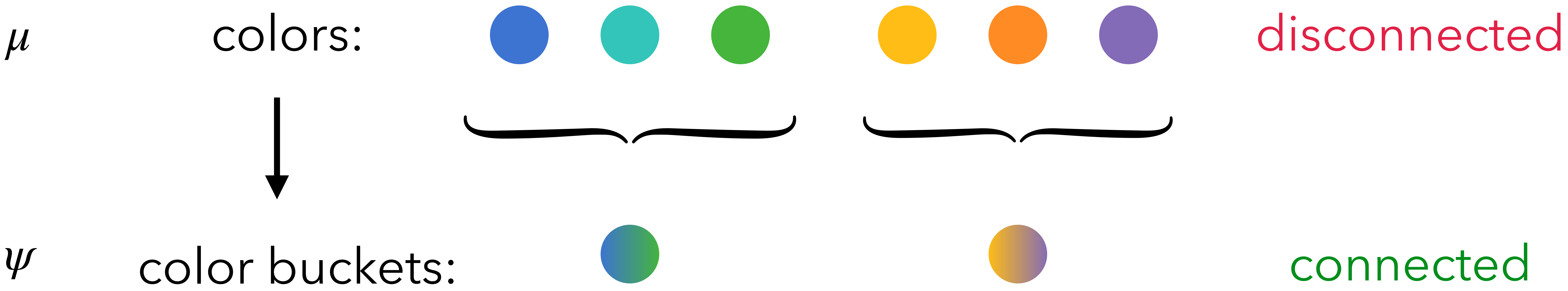
In [LWYY25], marginal measures are represented through a complex Markov chain.

$\mu_H(e$ is monochromatic$) = \sum$ percolation analysis on a convergent complex Markov chain

Hypergraph colorings may have disconnected state space.

Projection-lifting in classical sampling

State-compression: projected measure with connected state space [GLLZ18, FHY21].



Glauber dynamics mixes rapidly on the projected distribution ψ.

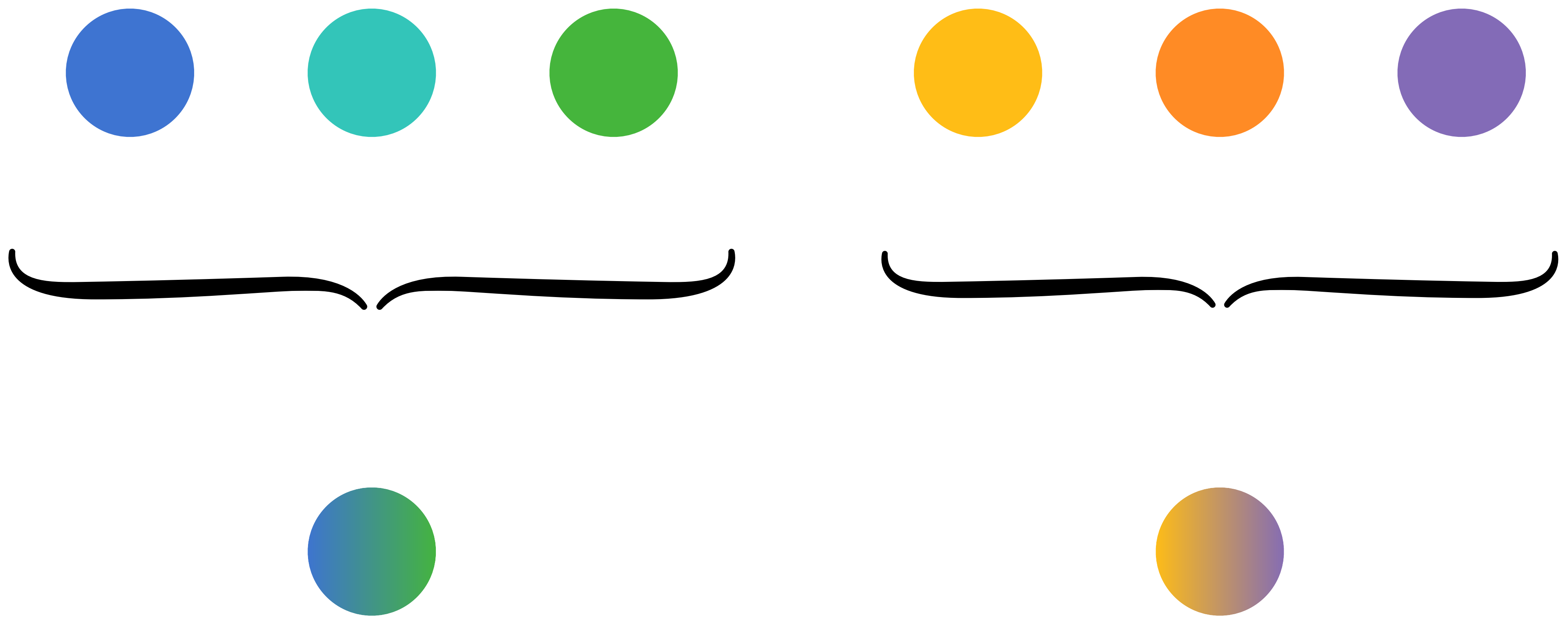
Projection-lifting in classical sampling

State-compression: projected measure with connected state space [GLLZ18, FHY21].

μ colors: disconnected

$\downarrow$

ψ color buckets: connected



Glauber dynamics mixes rapidly on the projected distribution ψ.

$$\psi \xrightarrow{\text{rejection sampling}} \mu$$

Projection-lifting in complex settings

To upper bound $|\mu(e \text{ is monochromatic})|$:

color buckets from ψ

complex Markov chain

$\xrightarrow{\text{extract}}$

color from μ

Projection-lifting in complex settings

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complex Markov chain

Proof idea:

(general complex measures may $\in \mathbb{C}$)

Design an event S on color buckets (ψ).

$\mu(e \text{ mono} \mid S) \in [0,1]$ is a probability

Complex Markov chain

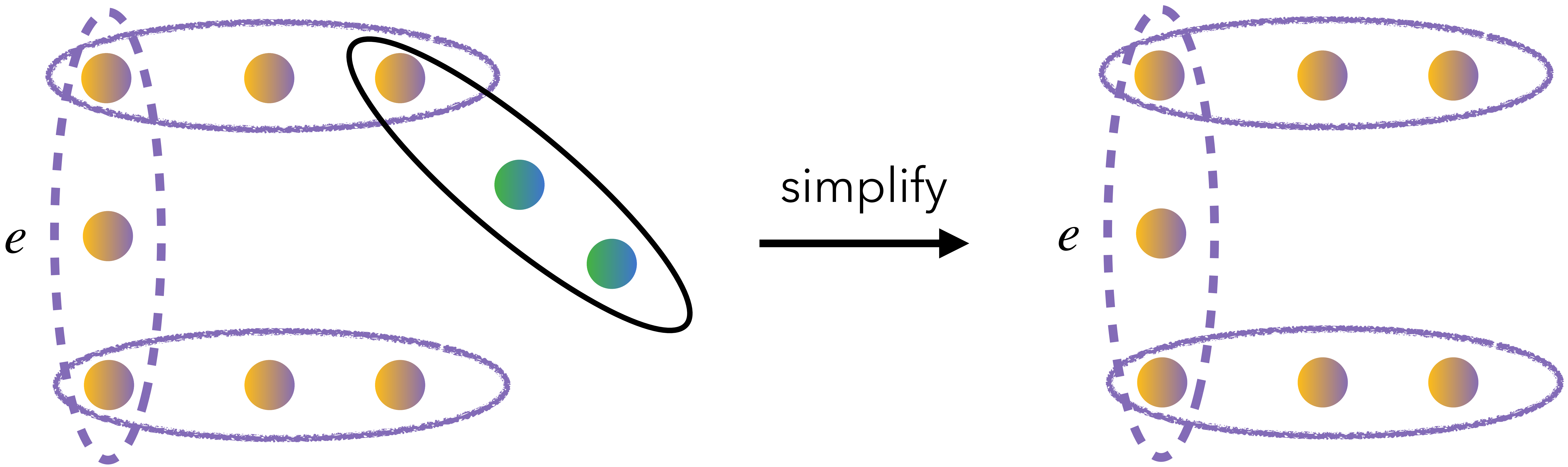
(triangle inequality) (conditional measure)

$$|\mu(e \text{ mono})| \le \sum_S \left|\mu(e \text{ mono} \wedge S)\right| = \sum_S \left|\mu(e \text{ mono} \mid S) \cdot \psi(S)\right| \le \sum_S \left|\psi(S)\right| < 1.$$

Projection-lifting in complex settings

Goal: design a event S on color buckets such that $\mu(e \text{ mono} \mid S) \in [0,1]$ is a probability.

Step 1: reveal bucket assignment on each vertex.



Bad cluster: maximal connected component containing hyperedge e with the same color bucket.

Event S: bad cluster and the color bucket.

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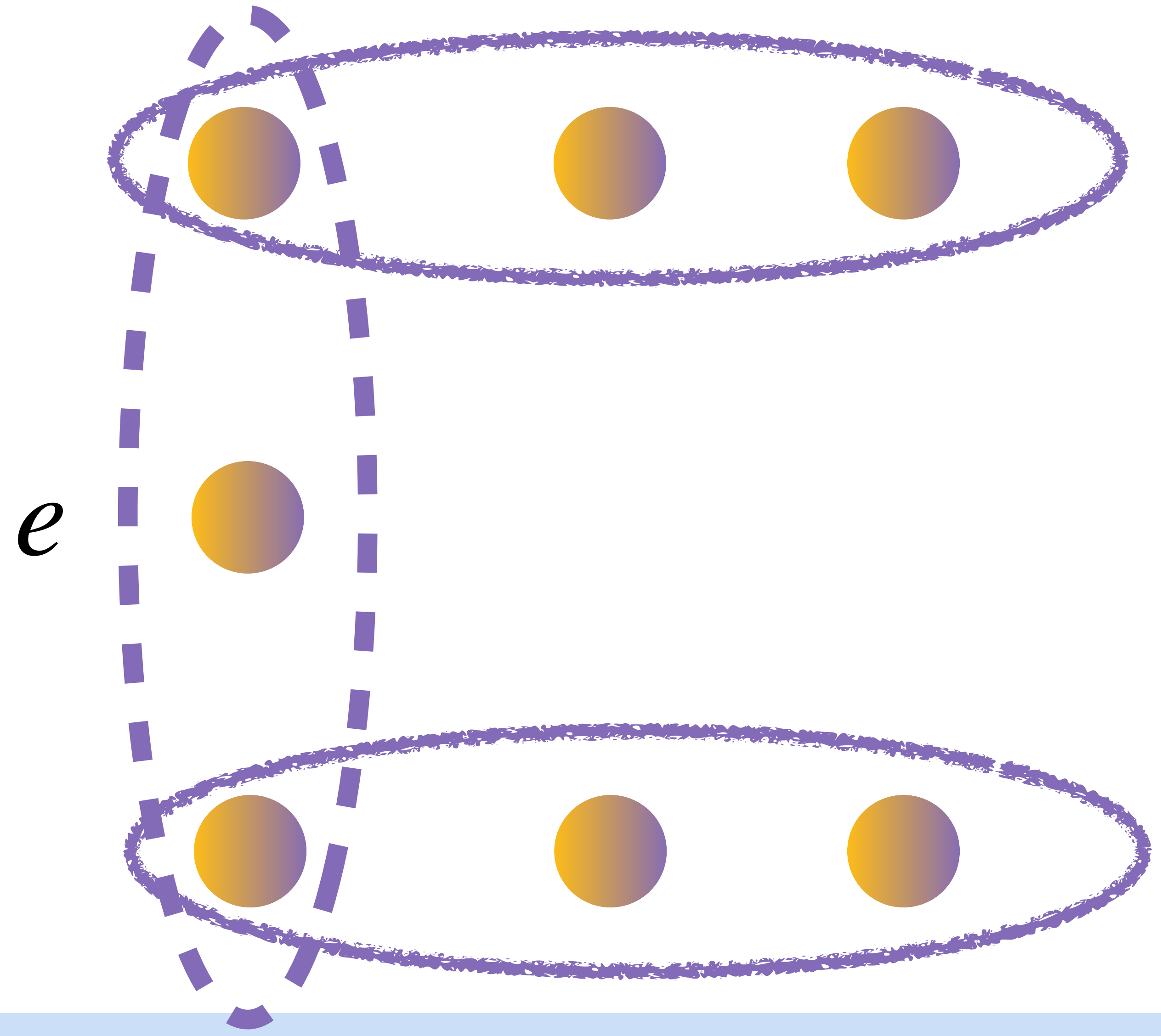

Event S: bad cluster and the color bucket.

$$\mu(e \text{ mono} \mid S) = \frac{\text{total complex weight for proper colorings consistent to } S \text{ with } e \text{ is monochromatic}}{\text{total complex weight for proper colorings consistent to } S}$$

Projection-lifting in complex settings

Goal: design a event S on color buckets such that $\mu(e \text{ mono} \mid S) \in [0,1]$ is a probability.

If the complex external field λ is on the color bucket, then we can extract the contributions from λ:

(S fixes the color bucket)

$$\mu(e \text{ mono} \mid S) = \frac{\lambda^X \cdot \text{number of proper colorings consistent with } S \text{ in which } e \text{ is monochromatic}}{\lambda^X \cdot \text{number of proper colorings consistent with } S}$$

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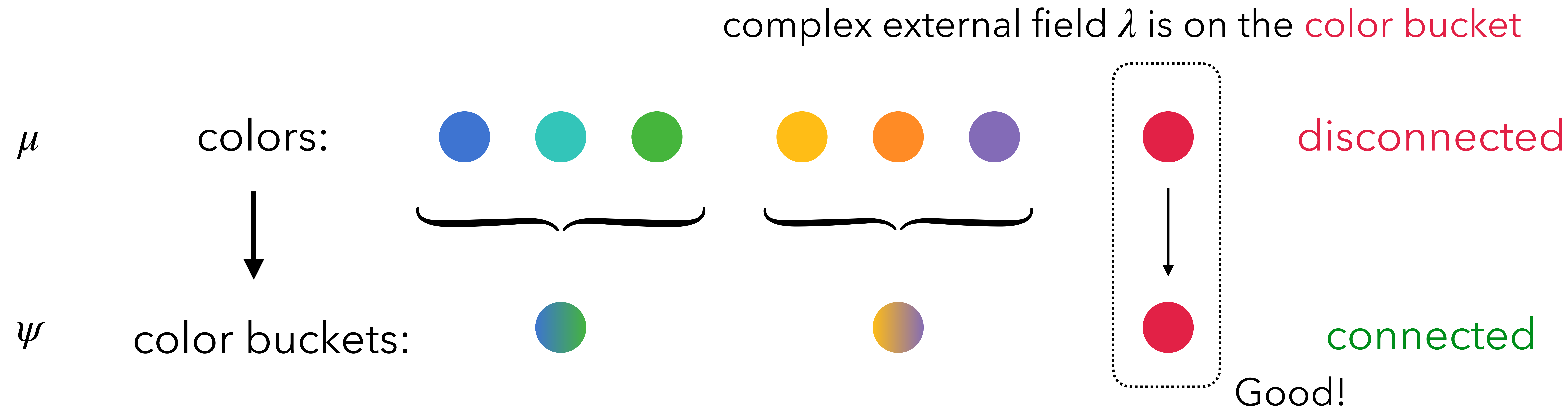
Step2: modified state-compression.



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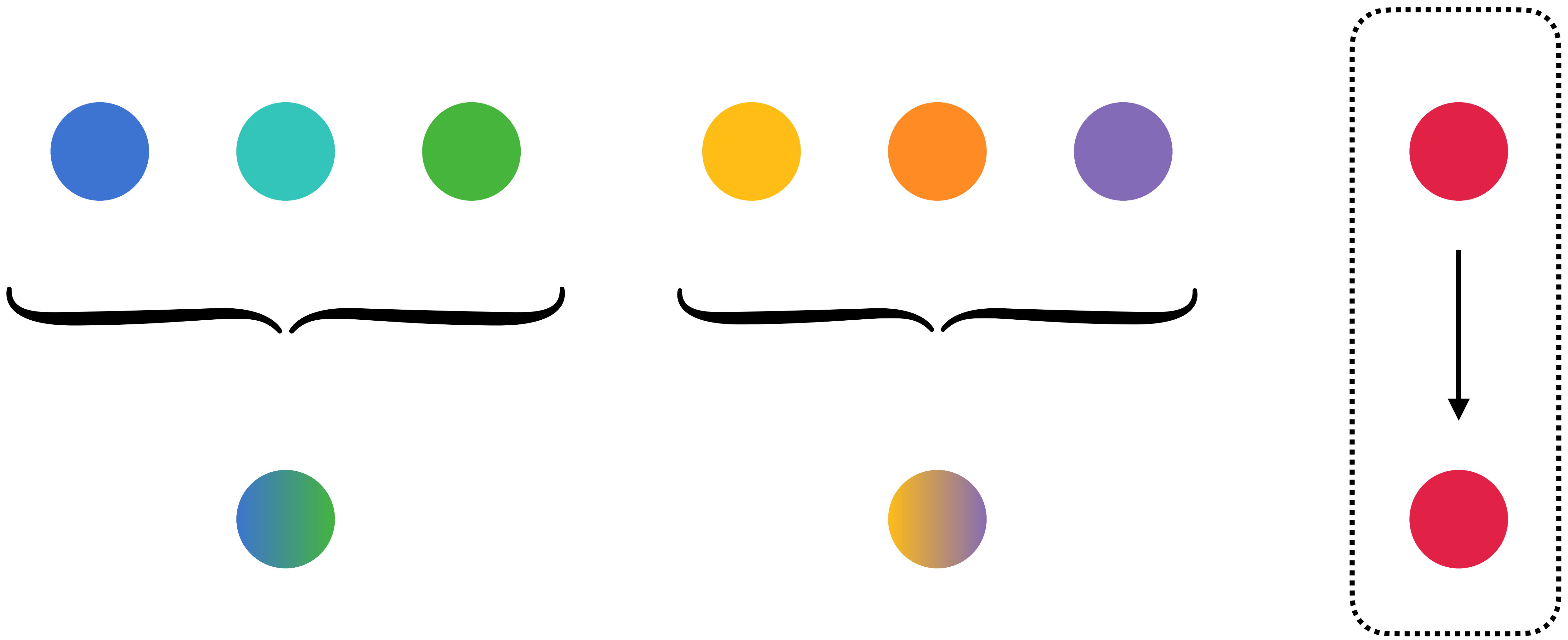
complex external field λ is on the color bucket

μ colors:

ψ color buckets:



disconnected

connected

Good!

This step is not free! It changes ψ.

Projection-lifting in complex settings

Proof idea:

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(triangle inequality) (conditional measure)

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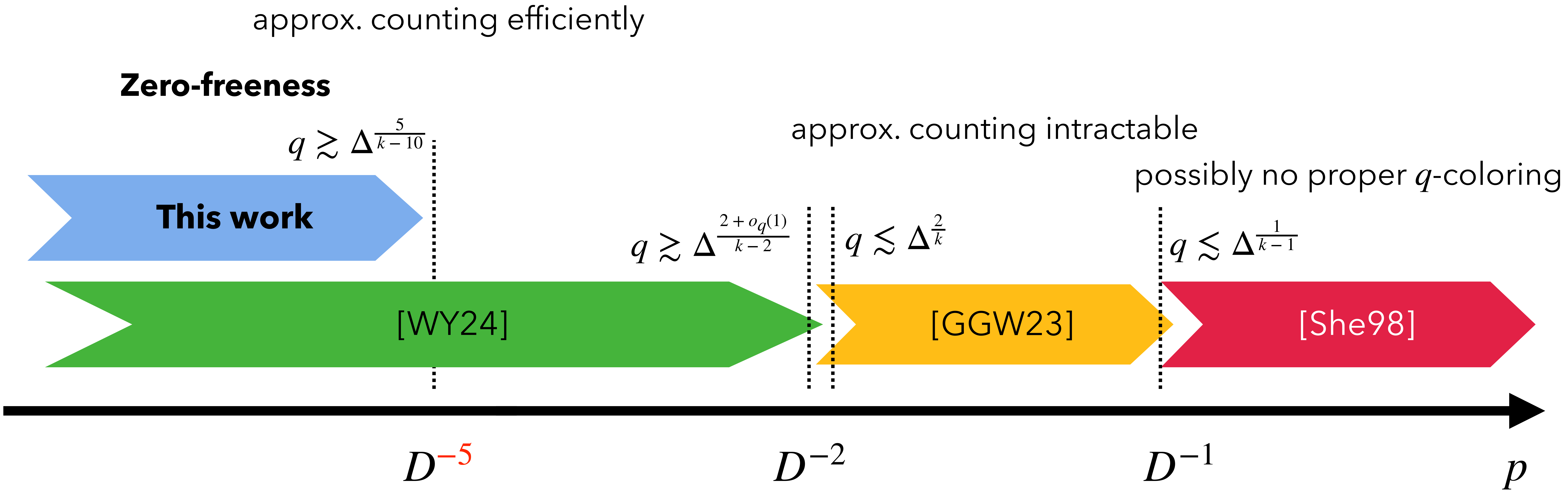
Refined analysis of the complex Markov chain on the modified ψ.

Summary and open problems



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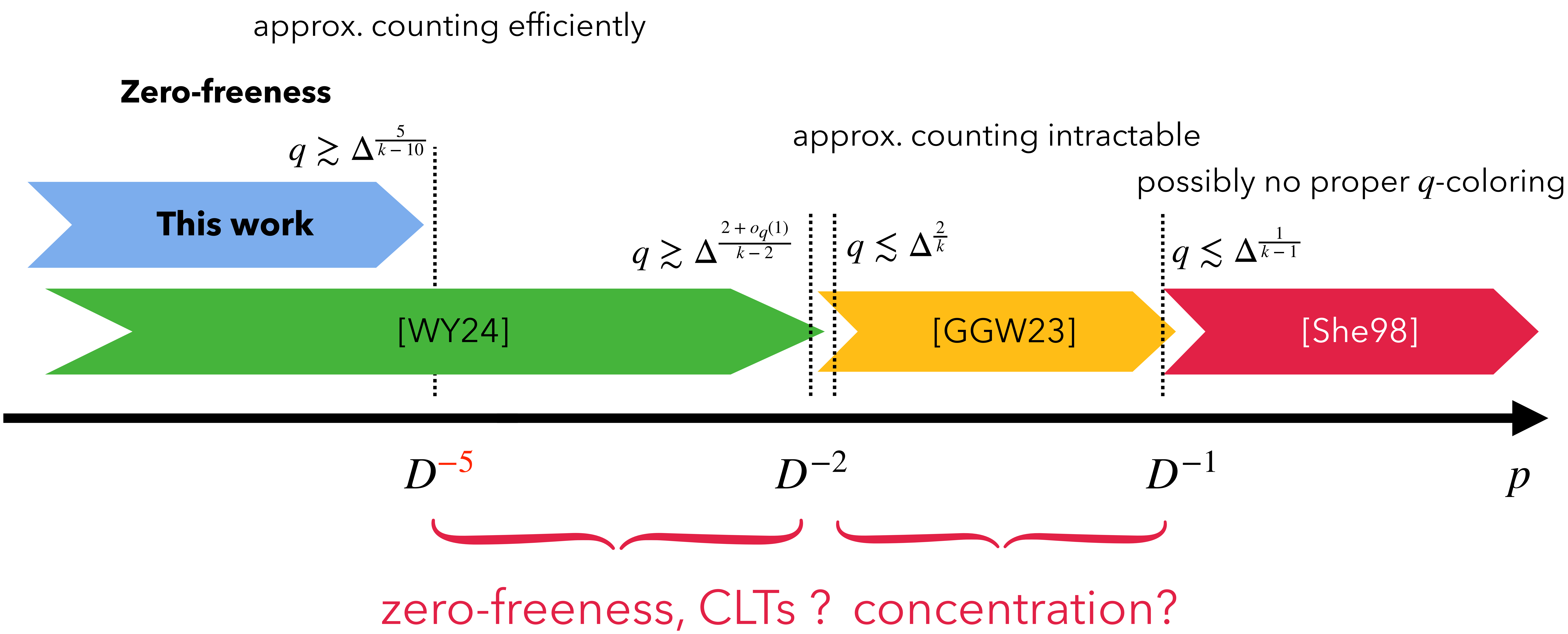

Thank you!