Phase Transitions via Complex Extensions of Markov Chains

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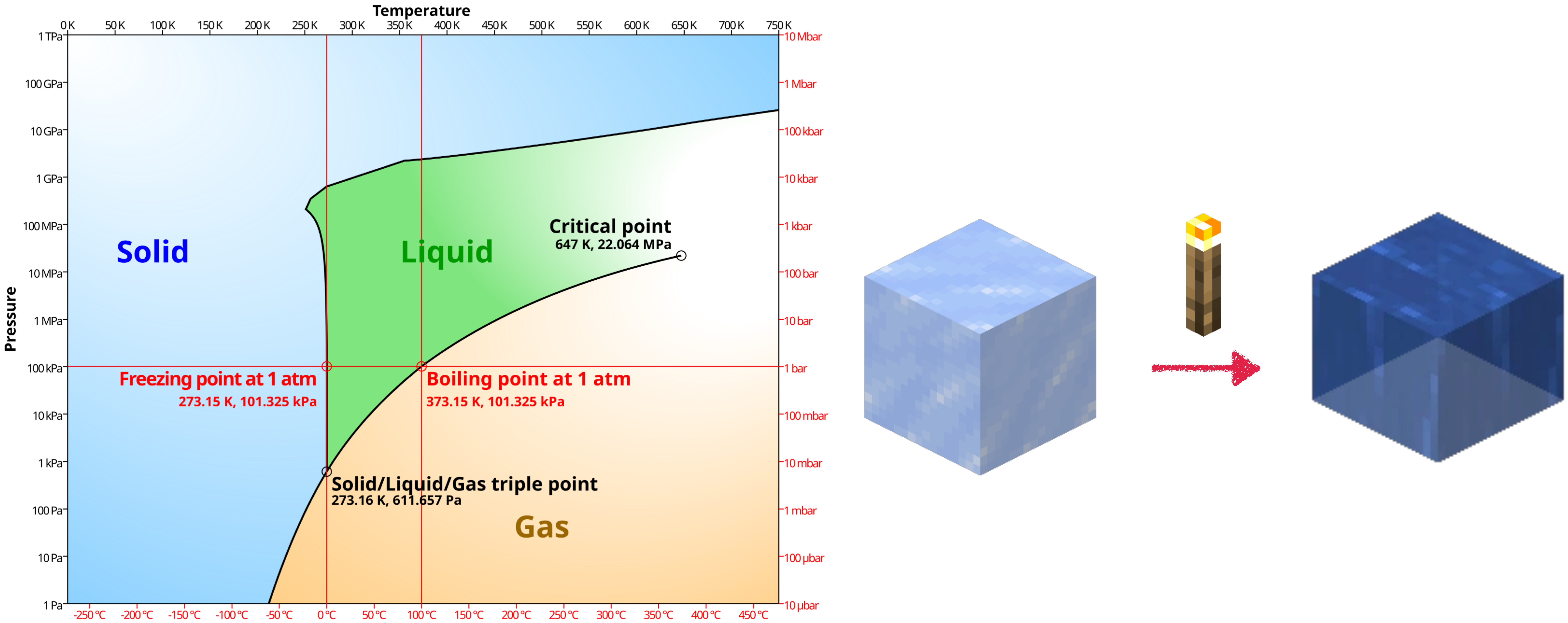
Joint work with Jingcheng Liu, Chunyang Wang and Yitong Yin

STOC 2025

Phase transition

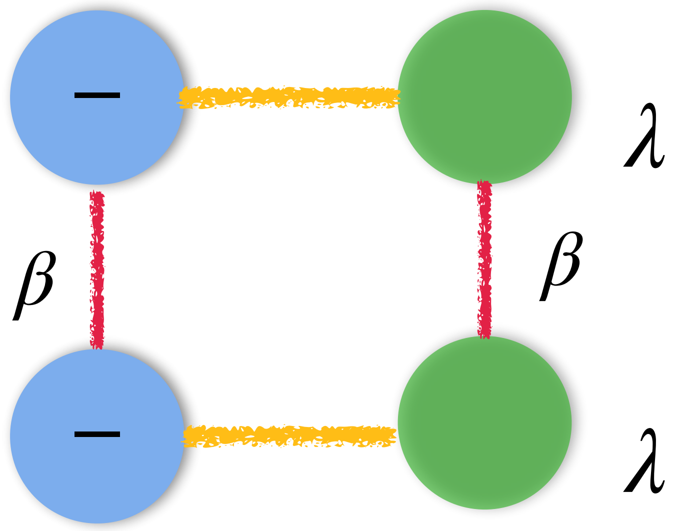


water’s phase transition

Phase transition and zero-freeness

Lee-Yang theory: phase transition ≈ complex zeros of partition function.

Example of zero-free region



Example of spin system

Computational phase transition - an example

Hardcore model

A graph $G = (V, E)$, a vertex weight $\lambda > 0$.

Ω: set of independent set.

Partition function $Z = \sum_{X \in \Omega} \lambda^{|X|}$. Gibbs distribution: $\forall X \in \Omega, \mu(X) = \frac{\lambda^{|X|}}{Z}$.

Computational phase transition - an example

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Approximately sample an independent set in μ.

Approximately compute the partition function Z.

(They are equivalent by [Jerrum, Valiant, Vazirani'86]).

Computational phase transition - an example

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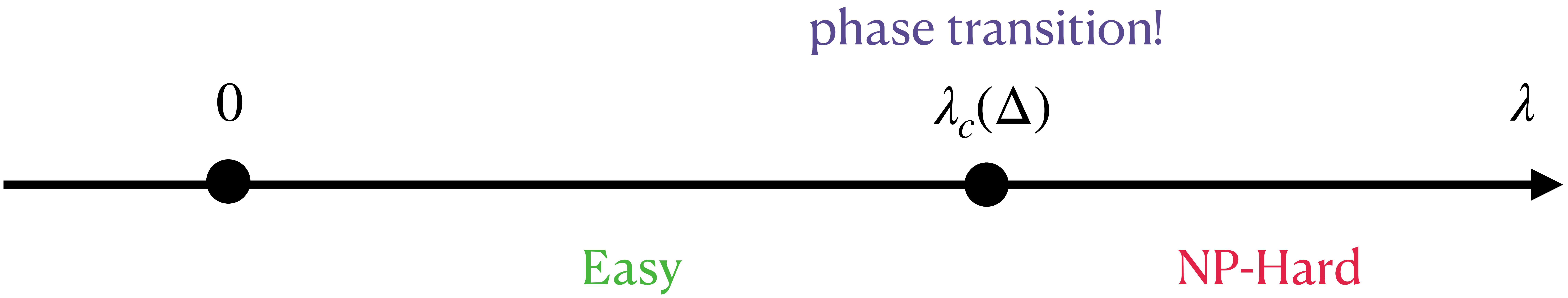

phase transition!

0

$\lambda_c(\Delta)$

λ

Easy

NP-Hard

Computational phase transition - an example

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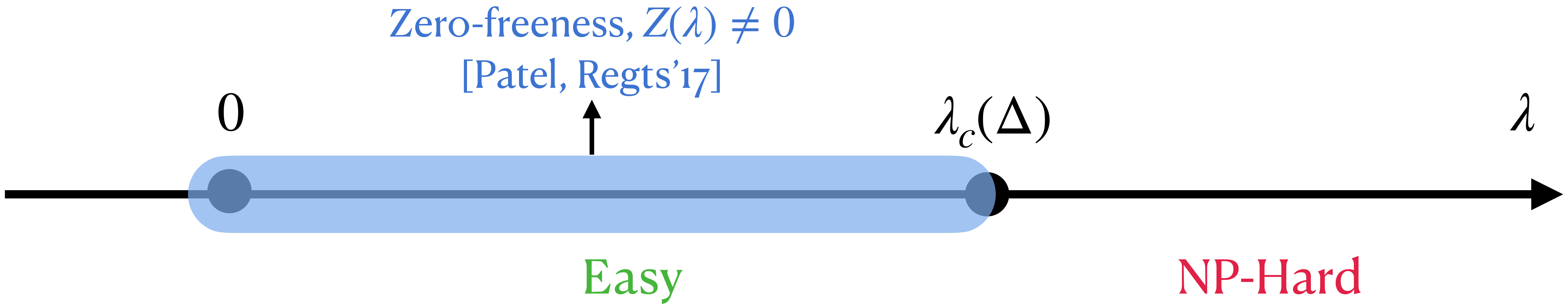

Zero-freeness, $Z(\lambda) \neq 0$
[Patel, Regts'17]

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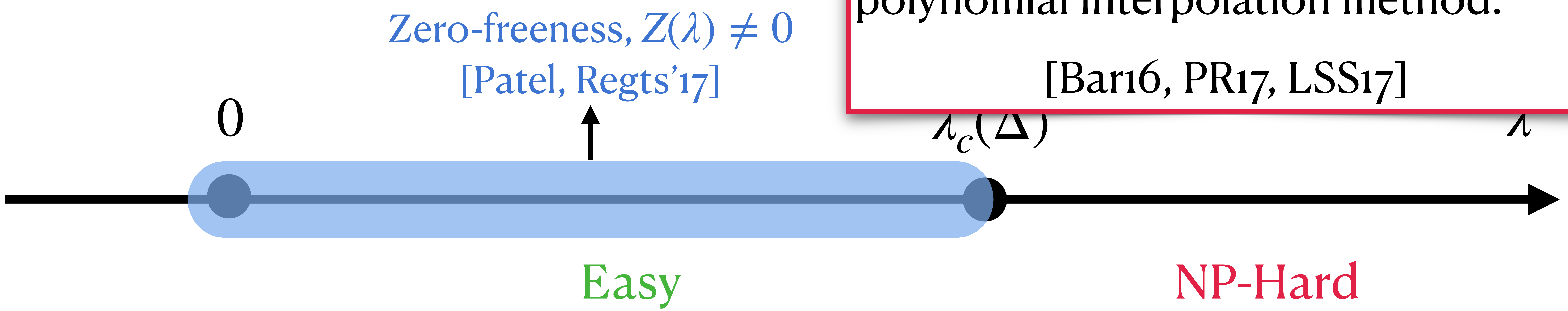
This implies an FPTAS by the polynomial interpolation method.

[Bar16, PR17, LSS17]



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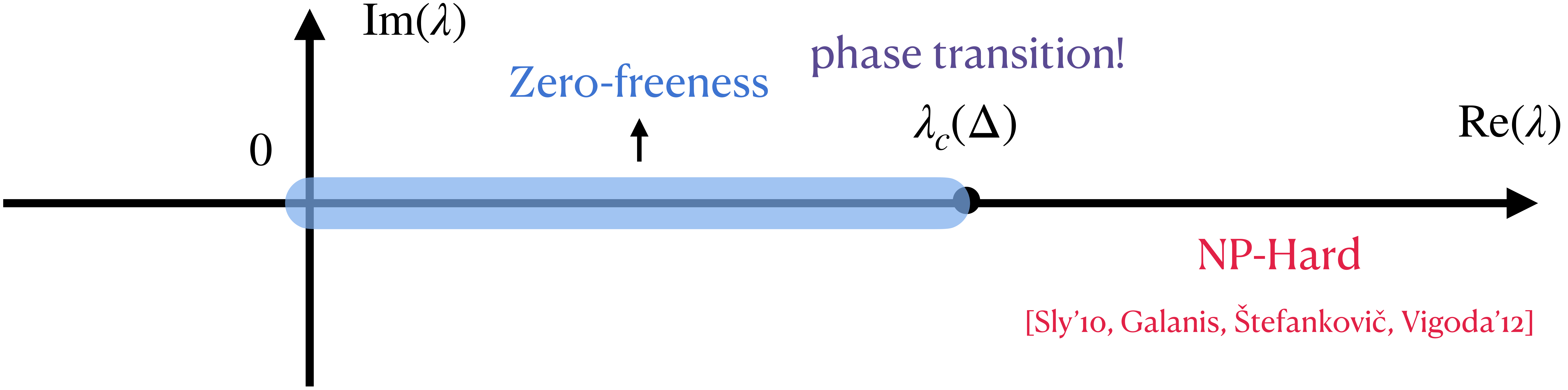

Different notions of phase transition matching $\lambda_c(\Delta)$:

Zero-freeness: [Patel, Regts'17]

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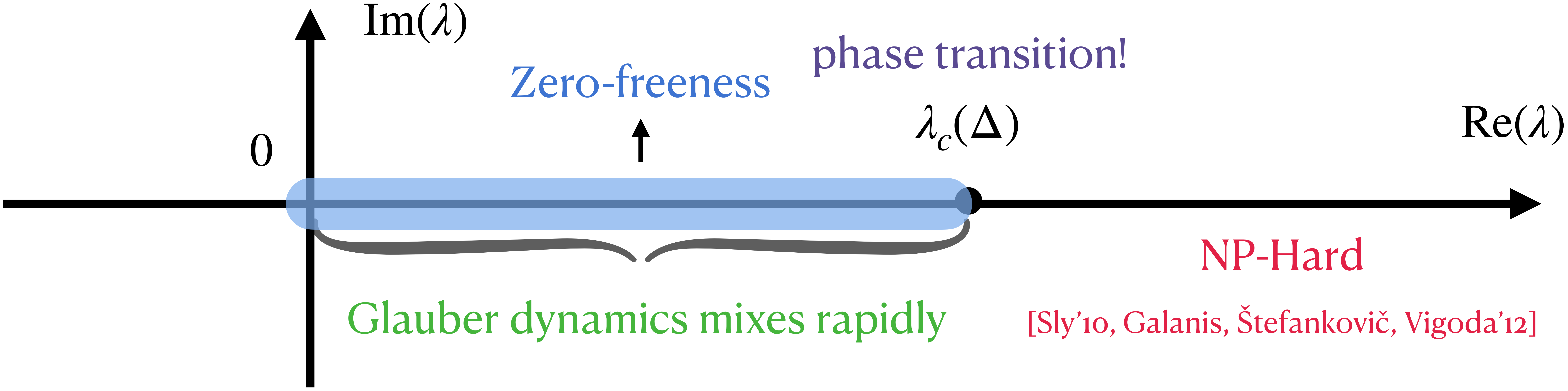

[Sly'10, Galanis, Štefankovič, Vigoda'12]

Different notions of phase transition matching $\lambda_c(\Delta)$:

Zero-freeness: [Patel, Regts'17]

Rapid mixing: [Chen, Liu, Vigoda'20, Chen, Liu, Vigoda'21, Chen, Feng, Yin, Zhang'22, Chen, Elden'22]

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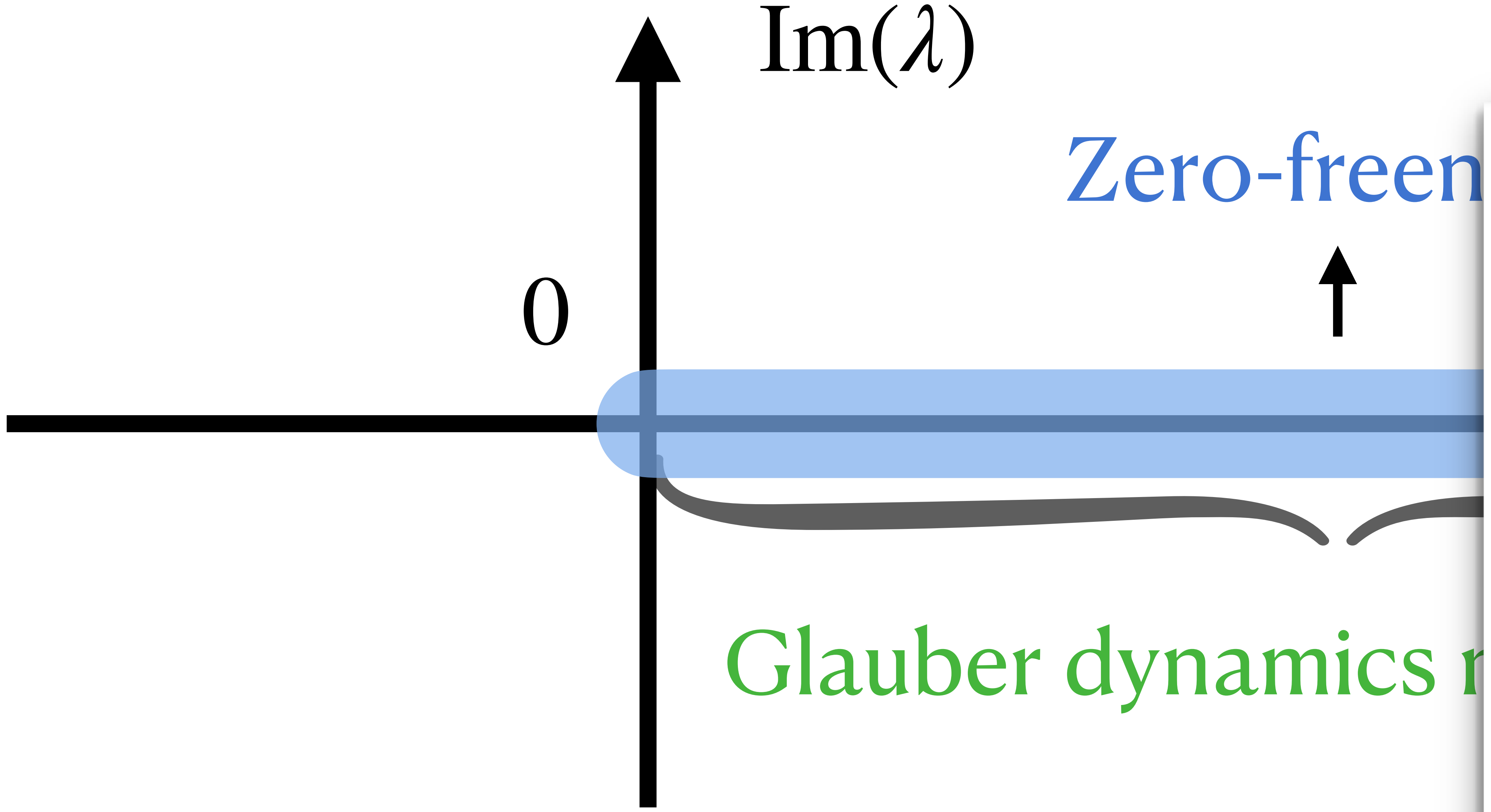
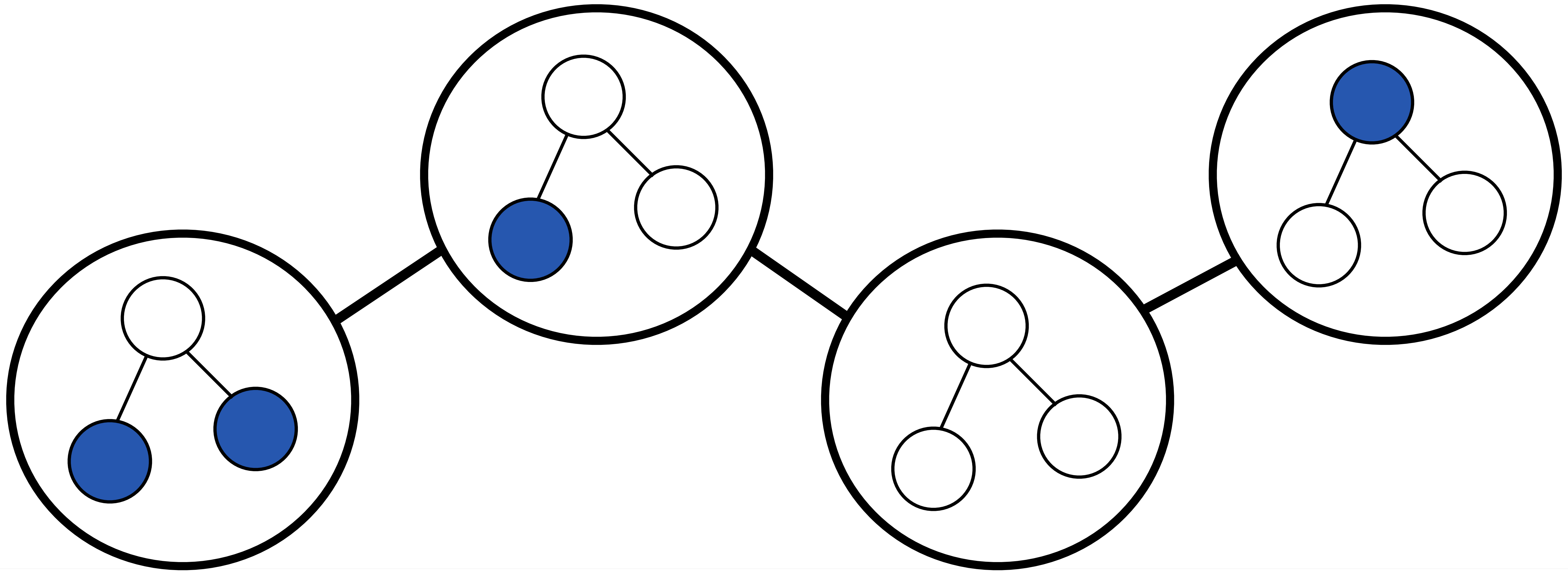


Rapid mixing: Glauber dynamics mixes rapidly



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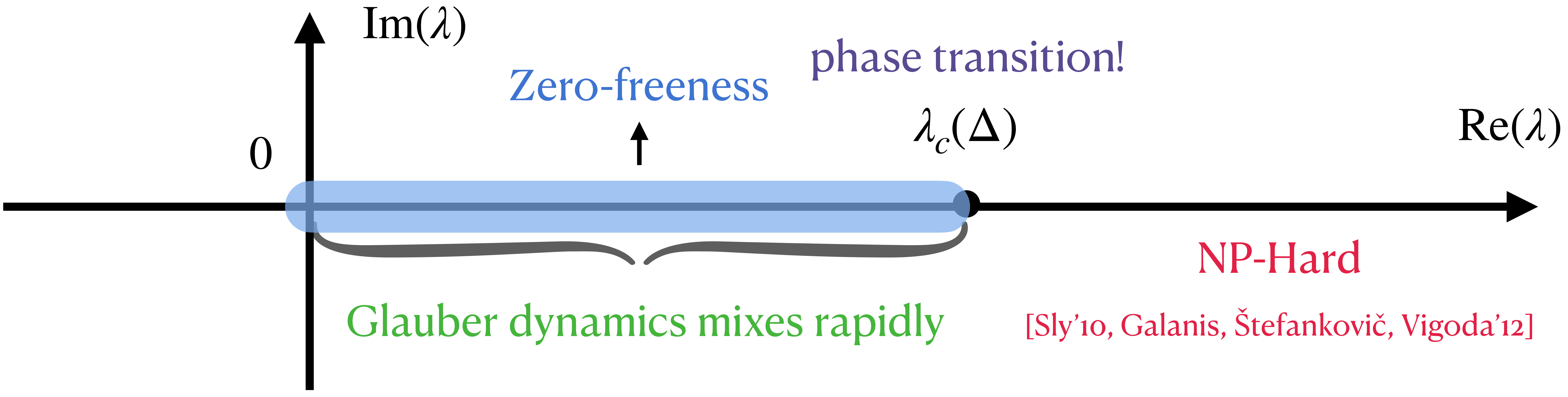

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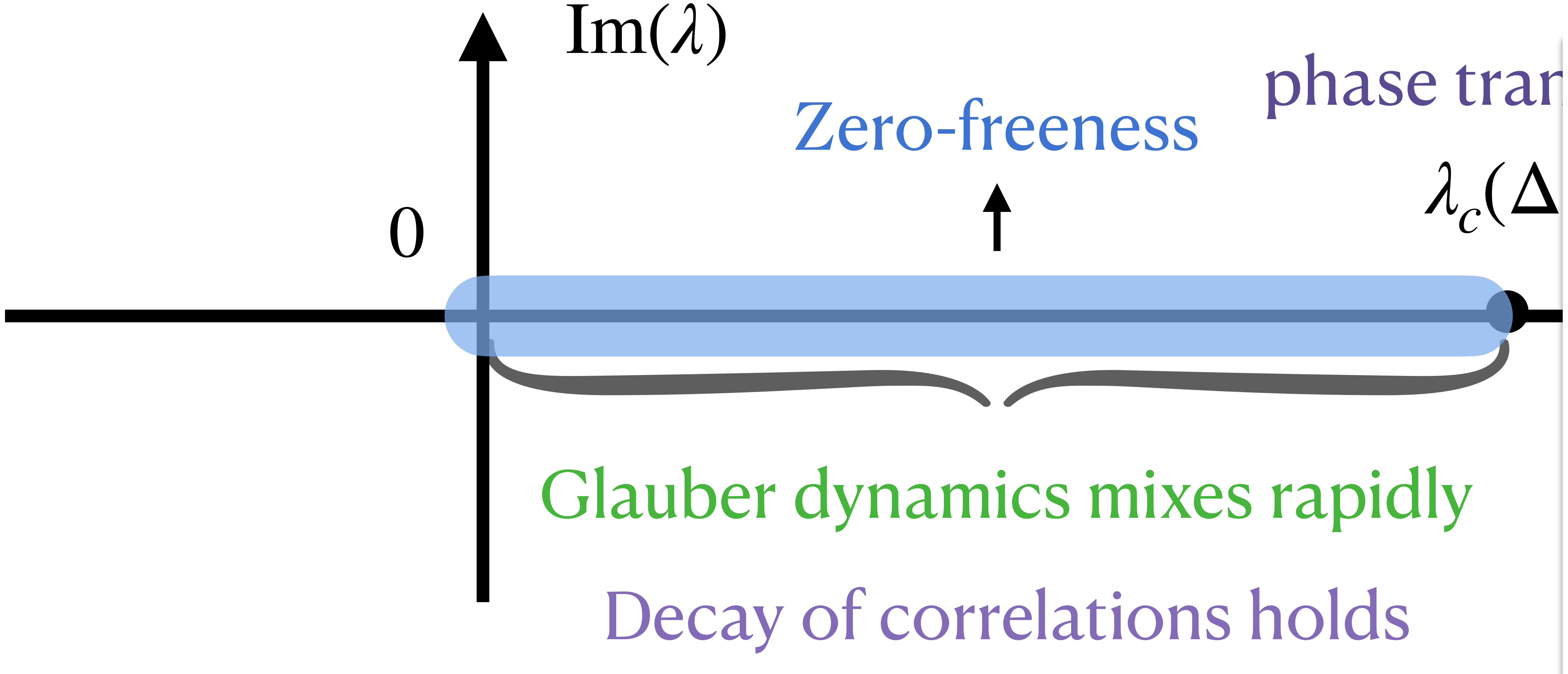
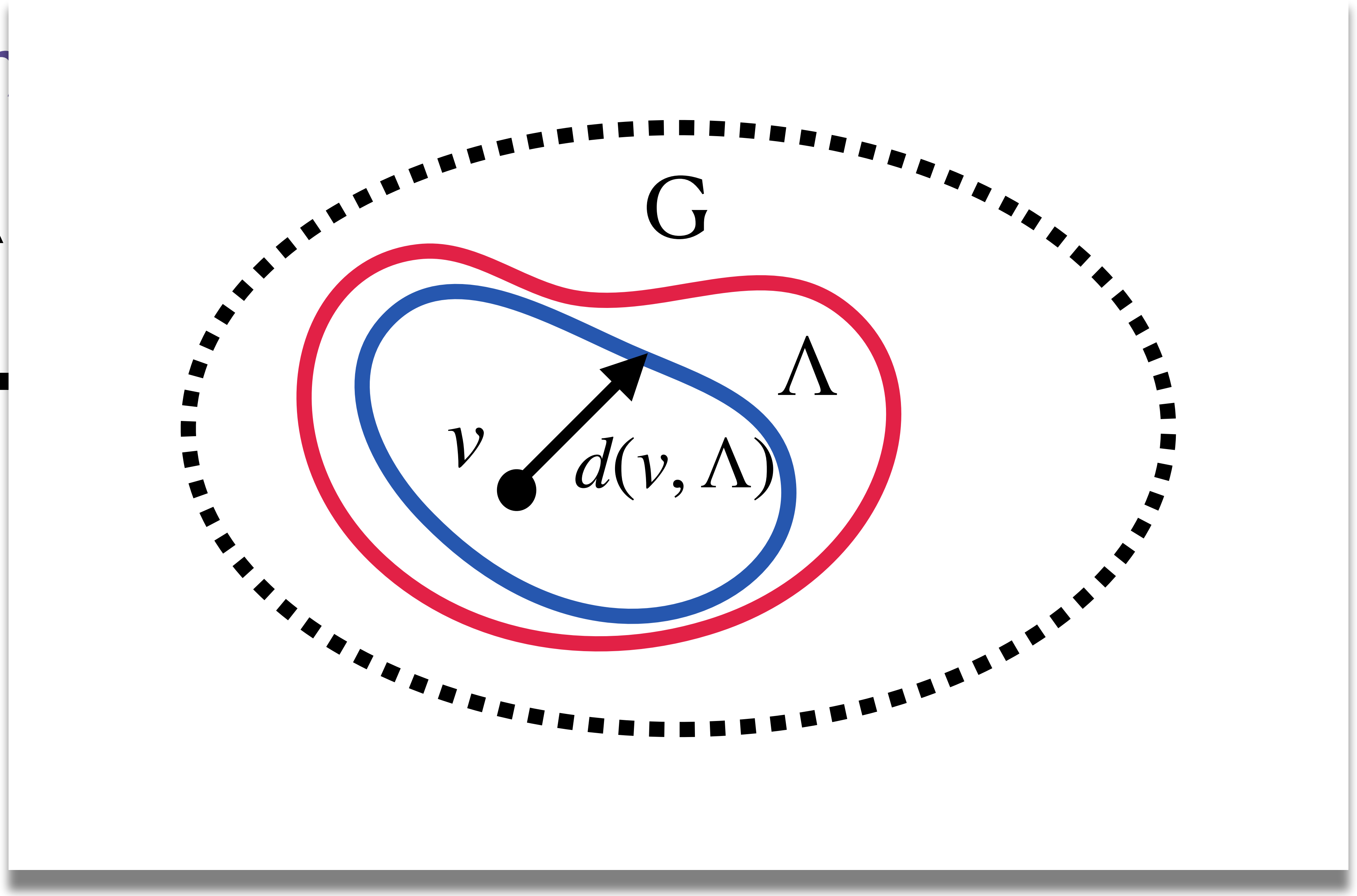


Zero-freeness

Glauber dynamics mixes rapidly

Decay of correlations holds



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Decay of correlations: [Weitz'06]

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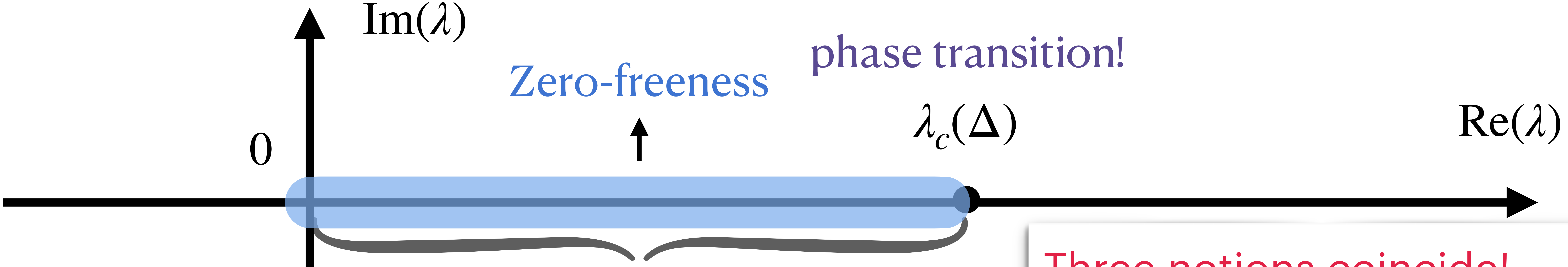

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Decay of correlations holds

Three notions coincide!
Are they equivalent?

Different notions of phase transition matching $\lambda_c(\Delta)$:

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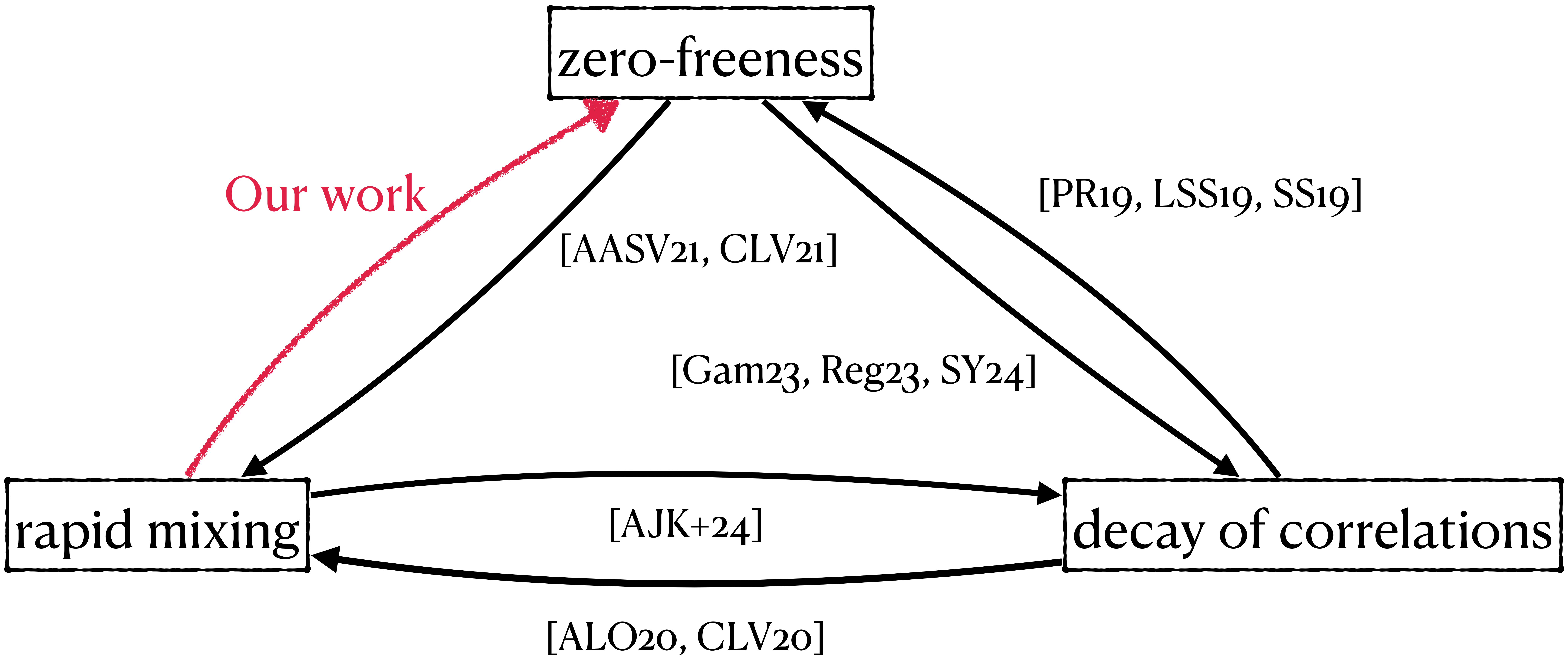
Decay of correlations: [Weitz'06]

Connections among three notions



Hypergraph independent set

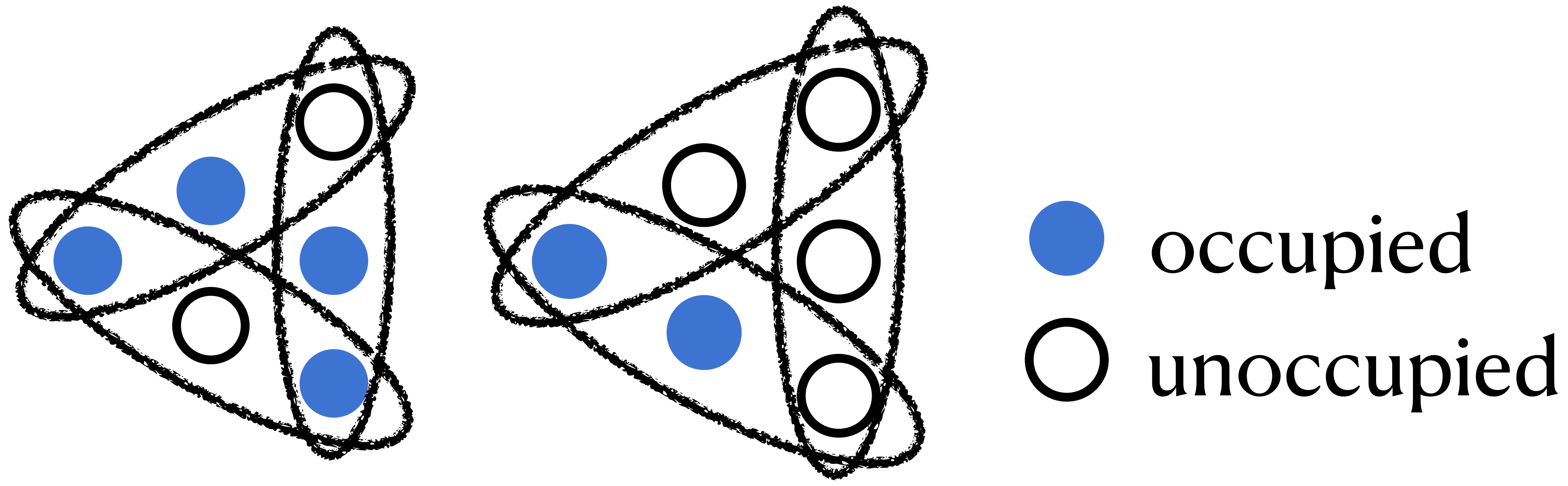
Hardcore model on hypergraph

A hypergraph $H = (V, \mathscr{E})$, a vertex weight $\lambda > 0$.

Ω set of hypergraph independent set.

Partition function $Z = \sum_{X \in \Omega} \lambda^{|X|}$.

Gibbs distribution: $\forall X \in \Omega, \mu(X) = \frac{\lambda^{|X|}}{Z}$.



Examples of hypergraph independent set

We consider the k-uniform hypergraph with maximum degree Δ.

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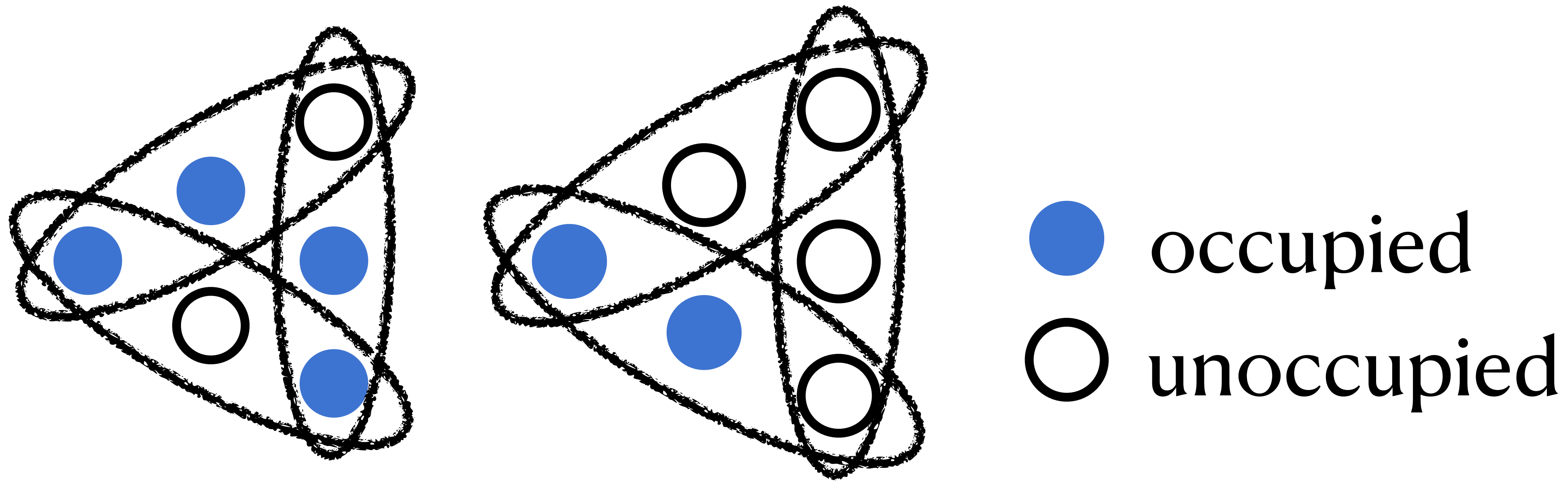


Examples of hypergraph independent set

We consider the k-uniform hypergraph with maximum degree Δ.

For $\lambda = 1$, Z is the number of HIS, μ is the uniform distribution of HIS.

Easy for $\Delta \lesssim 2^{k/2}$ ("sampling LLL condition") [HSZ19, HSW21, QWZ22, FGW+23].

NP-hard for $\Delta \geq 5 \cdot 2^{k/2}$ [BGG+19].

Rapid mixing of Markov chains

Approximate counting/sampling hypergraph independent sets under "sampling LLL conditions".

[Hermon, Sly, Zhang'19]: rapid mixing of Glauber dynamics.

[He, Sun, Wu'21, Qiu, Wang, Zhang'22]: perfect sampler.

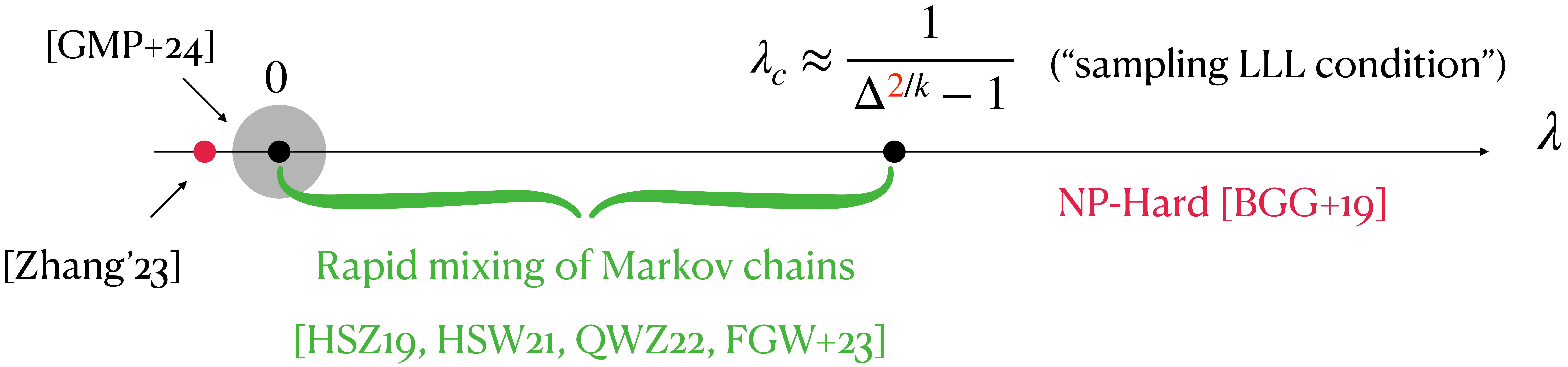
[Feng, Guo, Wang, Wang, Yin'23]: local sampler.

They are all based on Markov chains through the lens of percolation.

Zero-freeness

[Galvin, McKinley, Perkins, Sarantis, Tetali'24] shows a zero-free disk centered at origin with radius $\approx \frac{1}{e\Delta}$.

[Zhang'23] shows that for k-uniform linear hypergraph, there is a zero at $\lambda \approx -\frac{\log \Delta}{\Delta}$.



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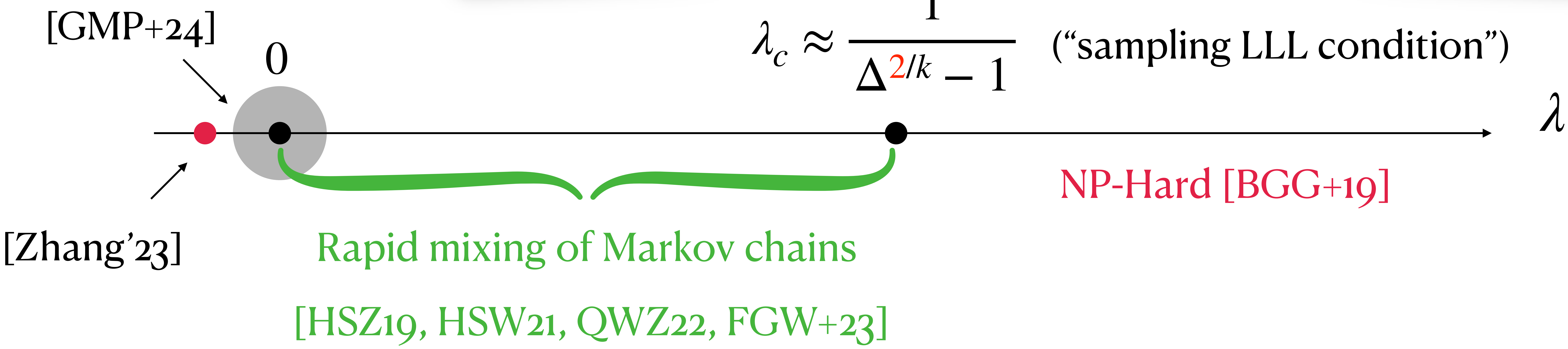
Zero-free region is lagging behind.

Existing tools for zero-free region can not capture the uniformity.



[GMP+24]

0

$\lambda_c \approx \frac{1}{\Delta^{2/k} - 1}$ ("sampling LLL condition")

λ

NP-Hard [BGG+19]

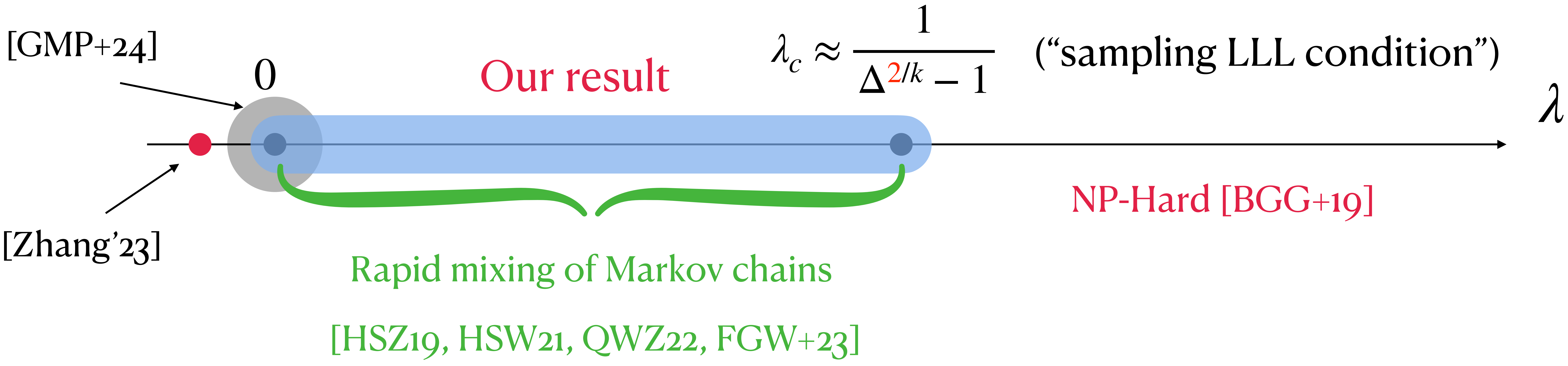
[Zhang'23]

Rapid mixing of Markov chains

[HSZ19, HSW21, QWZ22, FGW+23]

Our result - improved zero-free region from Markov chains

For *k*-uniform hypergraph with maximum degree Δ:



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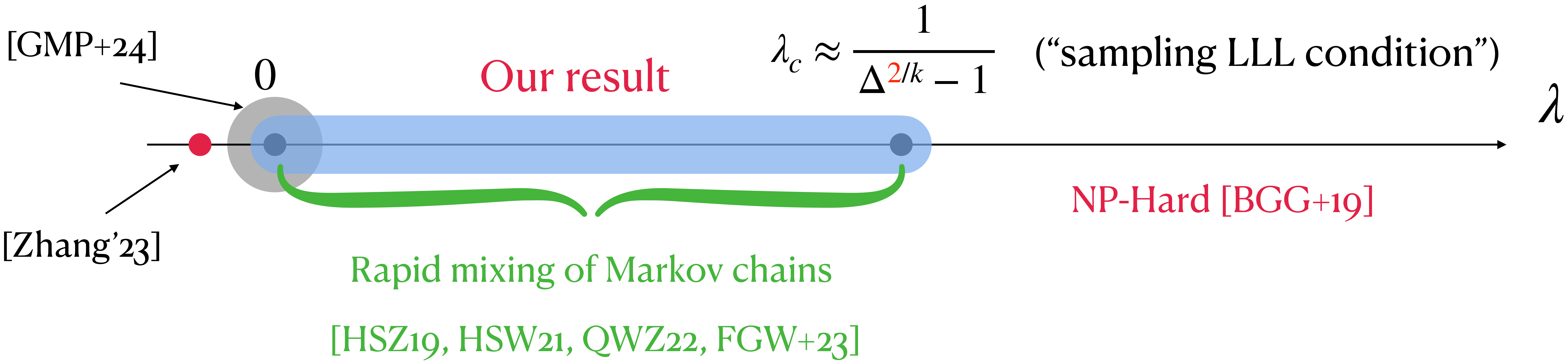

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NP-Hard [BGG+19]

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Corollaries of zero-freeness (in the same regime, informal):

1. FPTAS for approximating the partition function based on [Barvinok'16, Patel, Regts'17, Liu, Sinclair, Srivastava'17].
2. Central limit theorem and local central limit theorem based on [Michelen, Sahasrabudhe'19, Jain, Perkins, Sah, Sawhney'22].
3. FPTAS for approximating the number of t-size independent sets based on [Jain, Perkins, Sah, Sawhney'22].

Technical contribution - complex measure

A vertex weight $\lambda \in \mathbb{C} \backslash \{-1\}$.

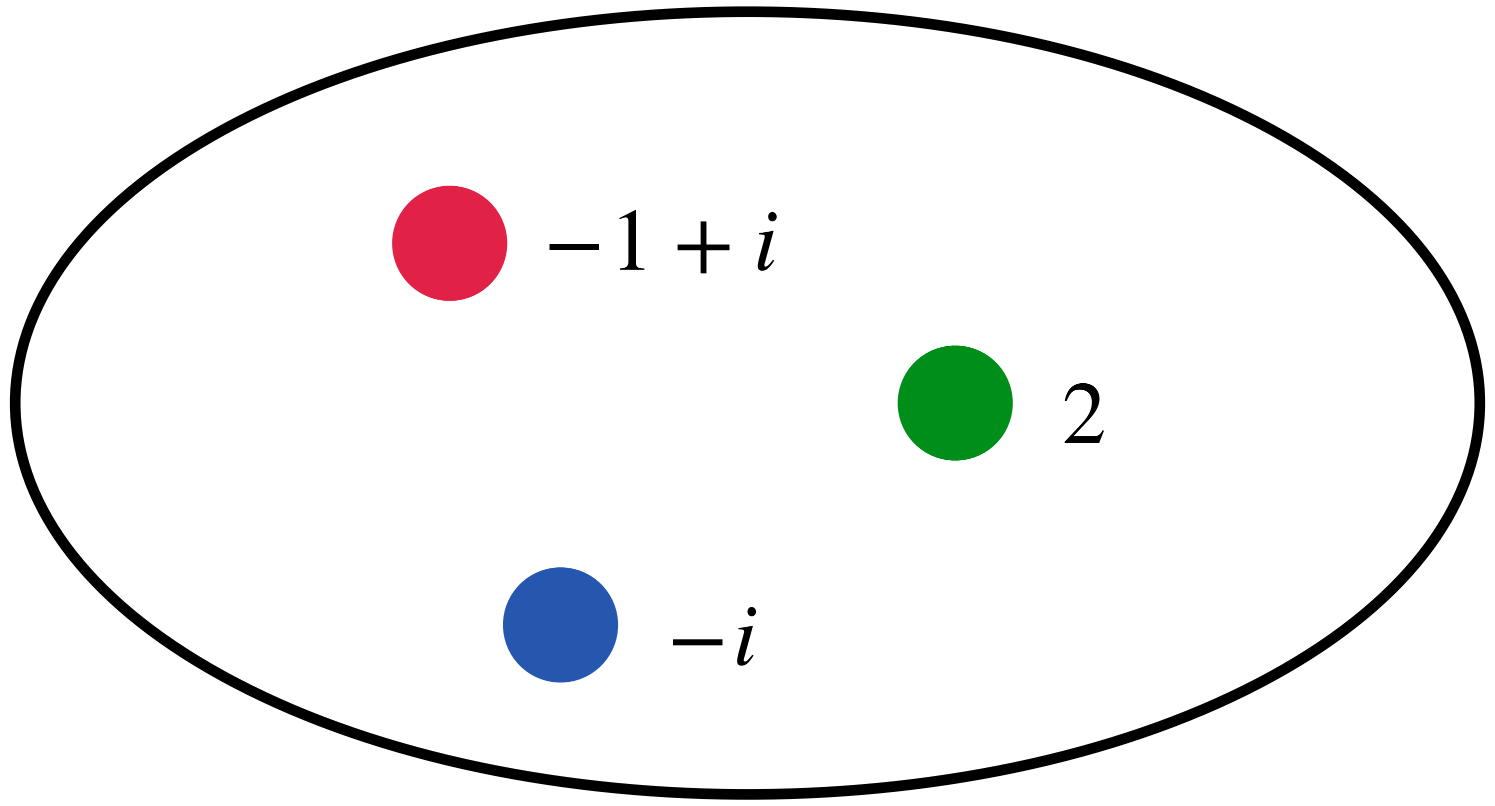
Ω = set of hypergraph independent sets.

Partition function $Z = \sum_{X \in \Omega} \lambda^{|X|}$.

Complex Gibbs measure: $\forall X \in \Omega, \mu(X) = \frac{\lambda^{|X|}}{Z}$.

We analyze complex Gibbs measure in a manner of distributions.

Technical contribution - complex measure



finite Ω

Normalized measure: $\mu(\Omega) = 1$.

Conditional measure: $\mu(\,\cdot \mid A) = \dfrac{\mu(\,\cdot \wedge A)}{\mu(A)}$ $(\mu(A) \neq 0)$.

Independence: $\mu(A_1 \cap A_2) = \mu(A_1) \cdot \mu(A_2)$.

Law of total measure: $\mu(B) = \sum_{i=1}^{m} \mu(B \cap A_i)$

(A_is are disjoint and $\bigcup_i A_i = \Omega$)

Complex measure μ over measurable space $(\Omega, \mathscr{F})$

Technical contribution - complex measure

For distributions, we have monotonicity:

For two events $B \subseteq A$, it holds that $\underset{\substack{\uparrow \\ \text{Hard to bound}}}{\mathbb{P}[B]} \leq \underset{\substack{\uparrow \\ \text{Easy to bound}}}{\mathbb{P}[A]}$.

For complex measure, monotonicity does not hold anymore!

Technical contribution - complex measure

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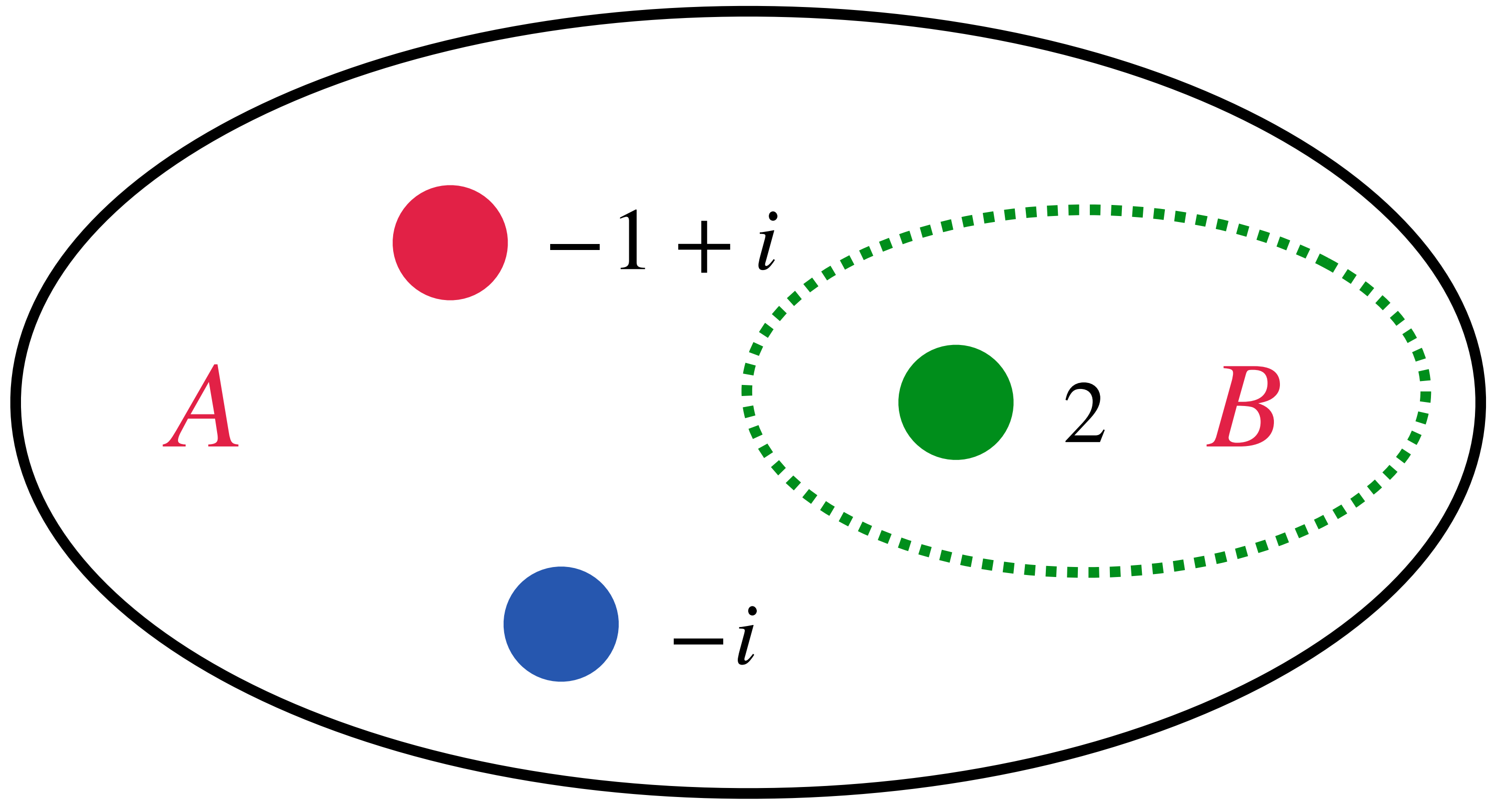
For two events $B \subseteq A$, it holds that $\mathbb{P}[B] \leq \mathbb{P}[A]$.

Hard to bound Easy to bound

For complex measure, monotonicity does not hold anymore!



$B \subseteq A$, but $\left|\mu(B)\right| > \left|\mu(A)\right|$

Technical contribution - complex measure

For complex measure, we use “zero-one law” to recover monotonicity.

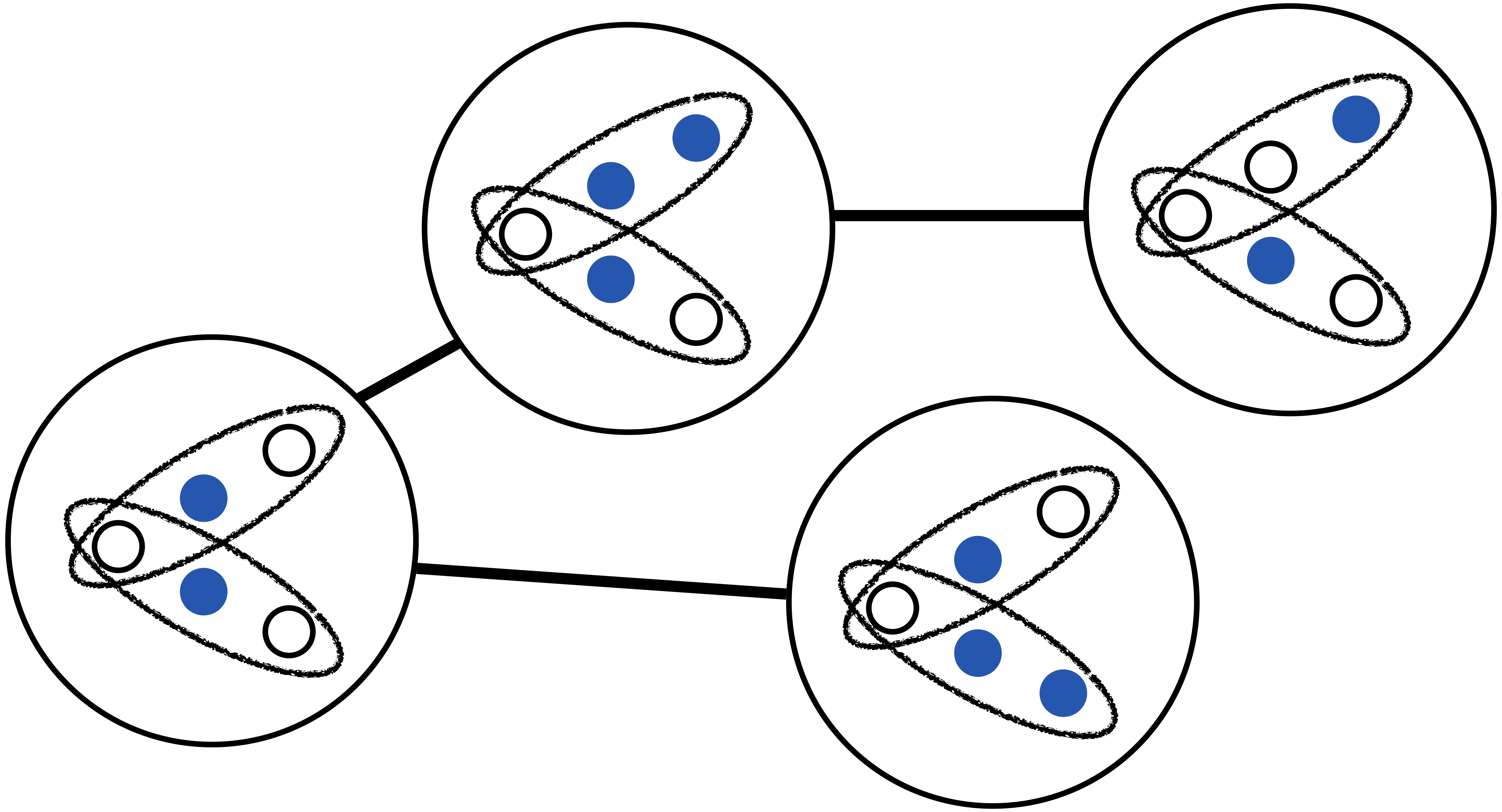
For two events $B \subseteq A$, it holds that

$$\left|\mu(B)\right| = \left|\mu(A \wedge B)\right| = \left|\mu(A)\right| \cdot \boxed{\left|\mu(B \mid A)\right|} \le \left|\mu(A)\right|.$$

A is a witness of B.

The key is to design a witness A, such that $\mu(B \mid A) \in \{0,1\}$ and $\left|\mu(A)\right|$ is easy to deal with.

Technical contribution - complex extensions of Markov chains



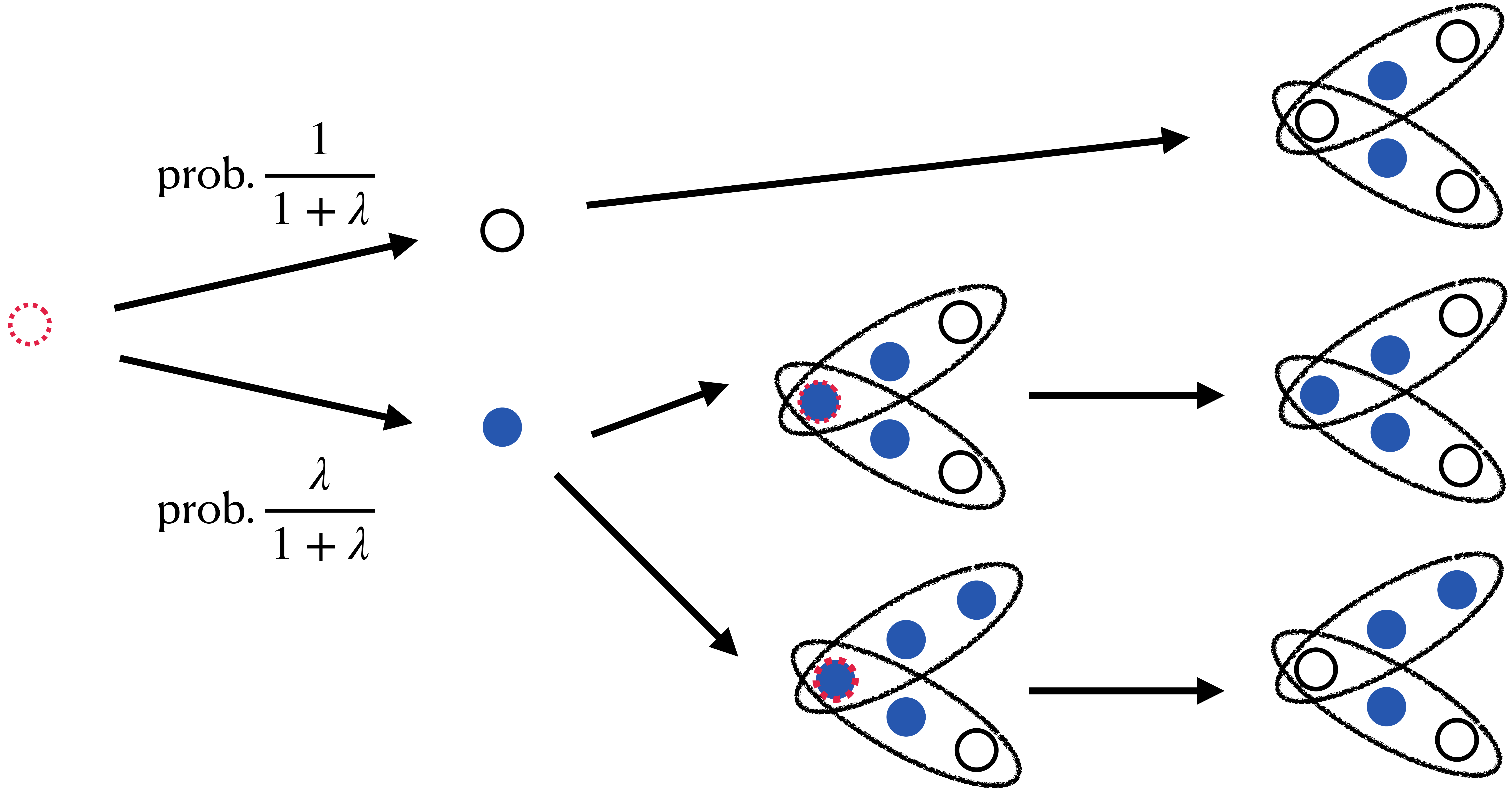
Start with an independent set.

In each update:

1. Choose a vertex *v* u.a.r.;
2. Update *v*.

Classical Glauber dynamics

Technical contribution - complex extensions of Markov chains



prob. $\frac{1}{1+\lambda}$

prob. $\frac{\lambda}{1+\lambda}$

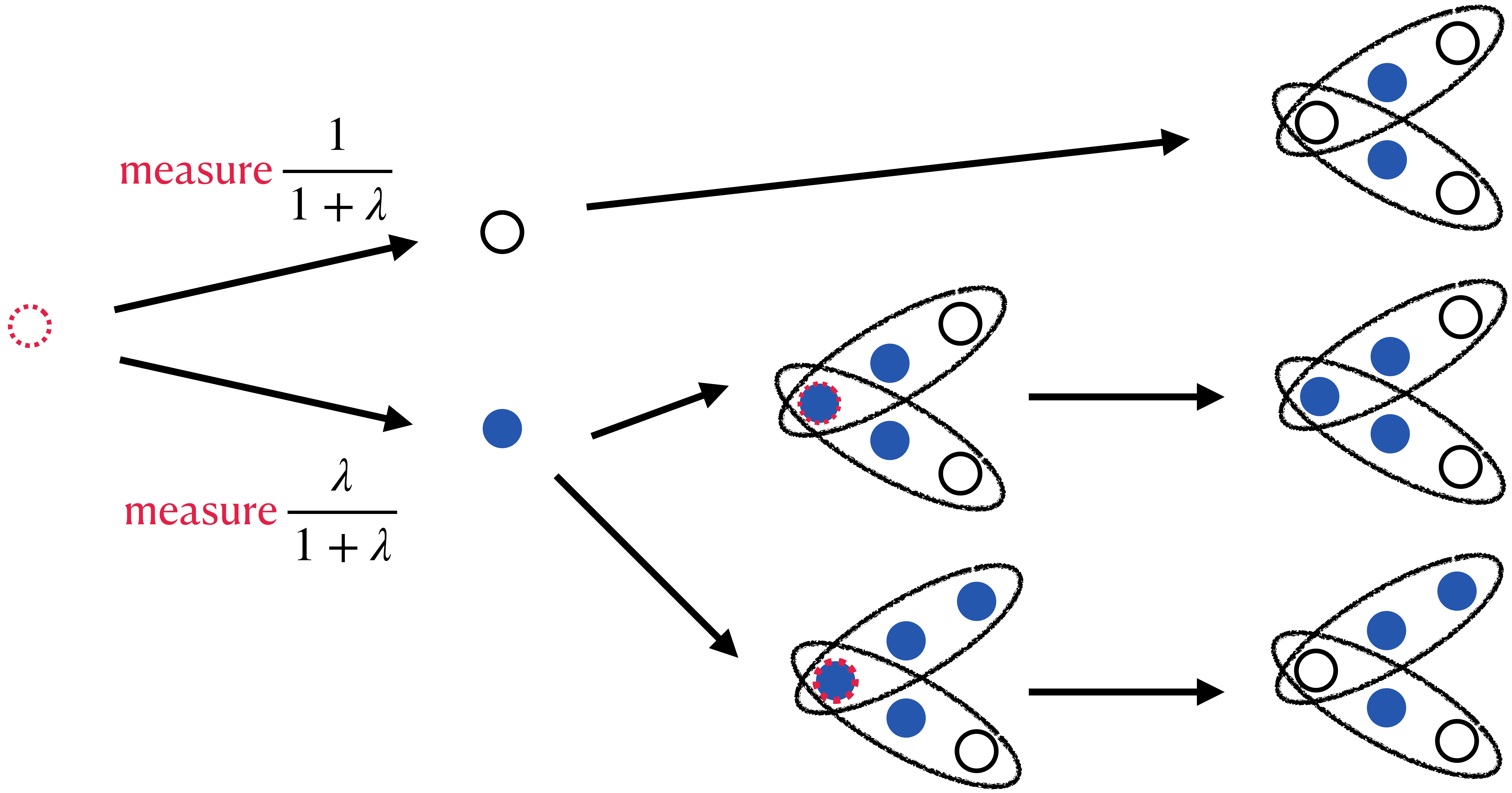
Propose

Check neighbors

Final decision

Update rule of classical Glauber dynamics

Technical contribution - complex extensions of Markov chains



measure $\frac{1}{1+\lambda}$

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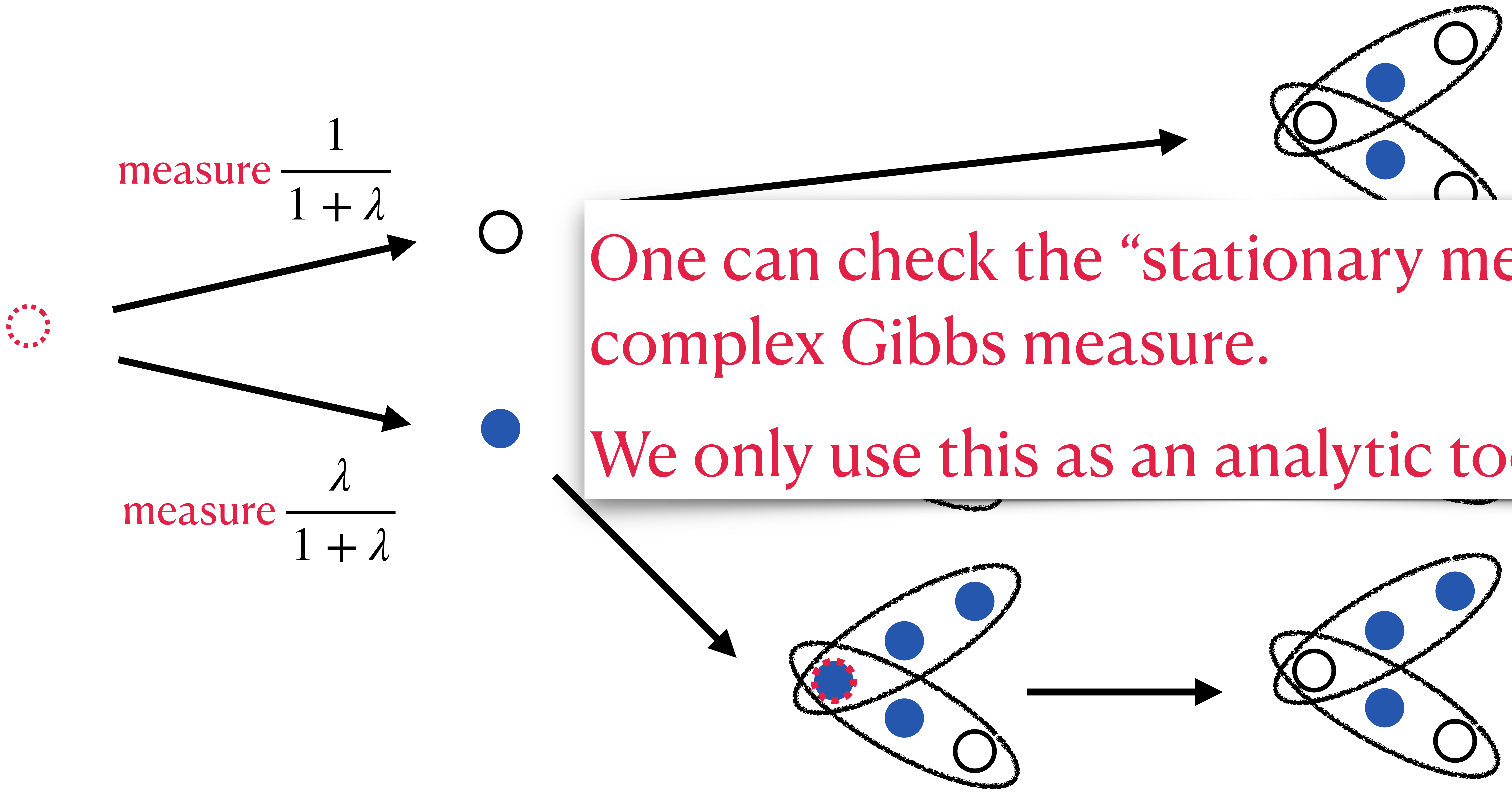
Final decision

Update rule of complex Glauber dynamics

Technical contribution - complex extensions of Markov chains



measure $\frac{1}{1+\lambda}$

measure $\frac{\lambda}{1+\lambda}$

One can check the "stationary measure" is the complex Gibbs measure.

We only use this as an analytic tool.

Propose

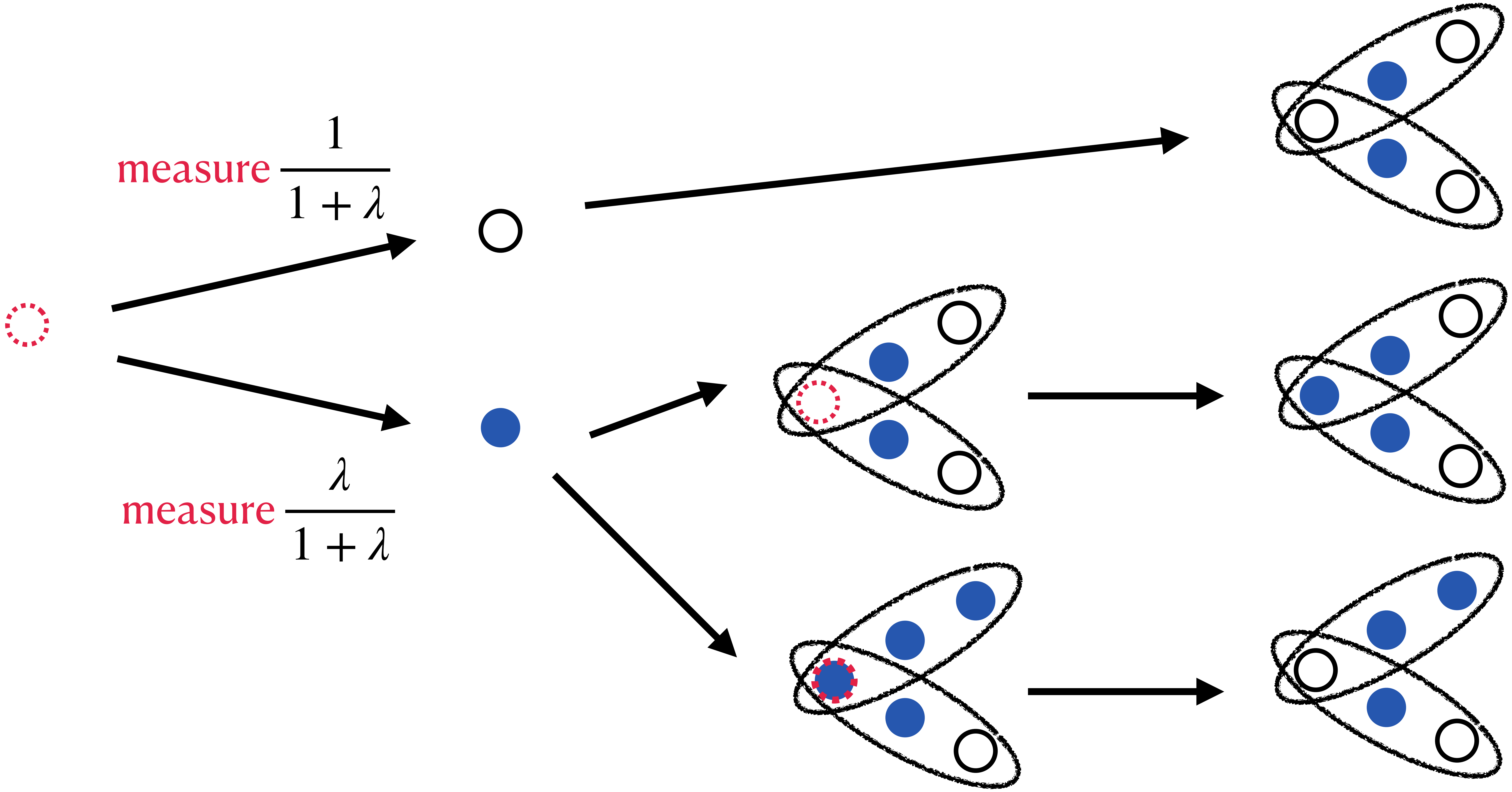
Check neighbors

Final decision

Update rule of complex Glauber dynamics

Technical contribution - complex percolation

Decomposition: decompose each transition into oblivious part and non-oblivious part.



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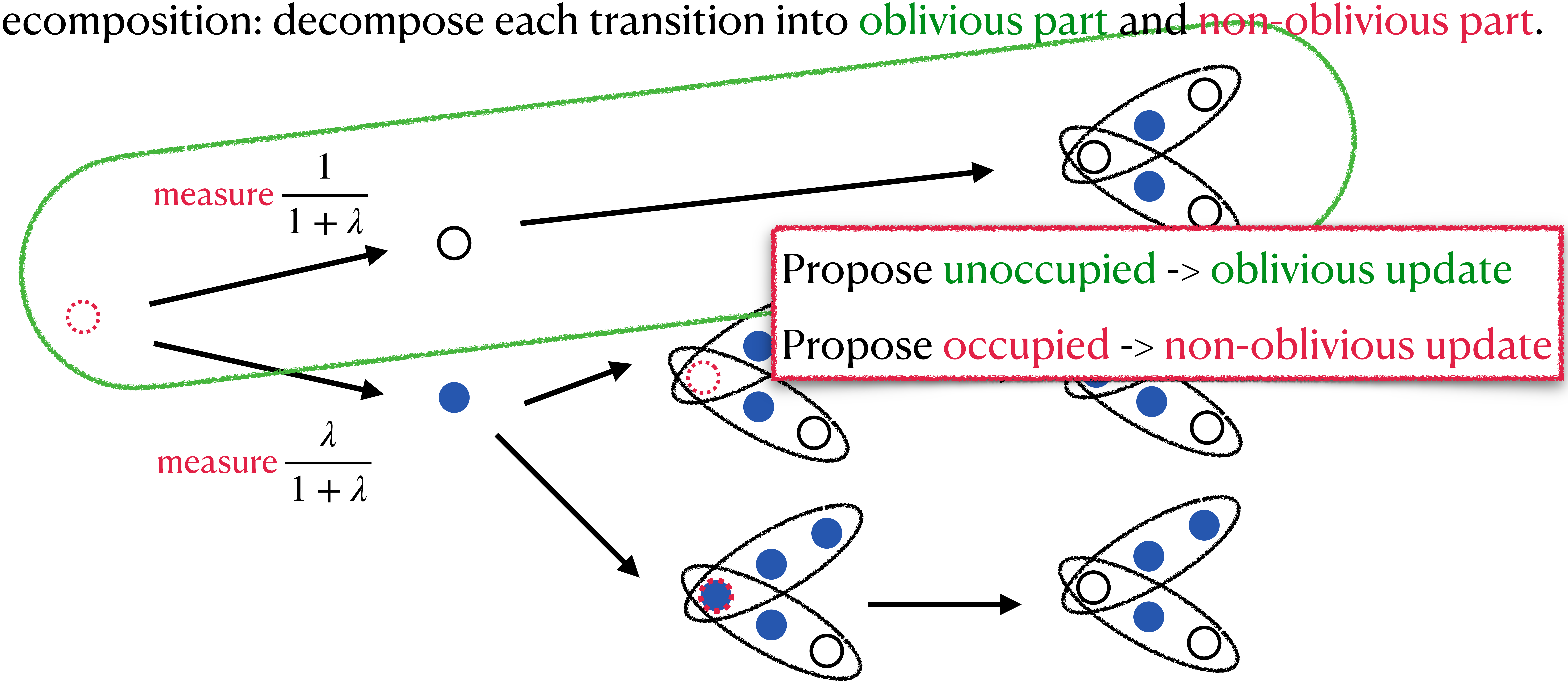

Propose unoccupied -> oblivious update

Propose occupied -> non-oblivious update

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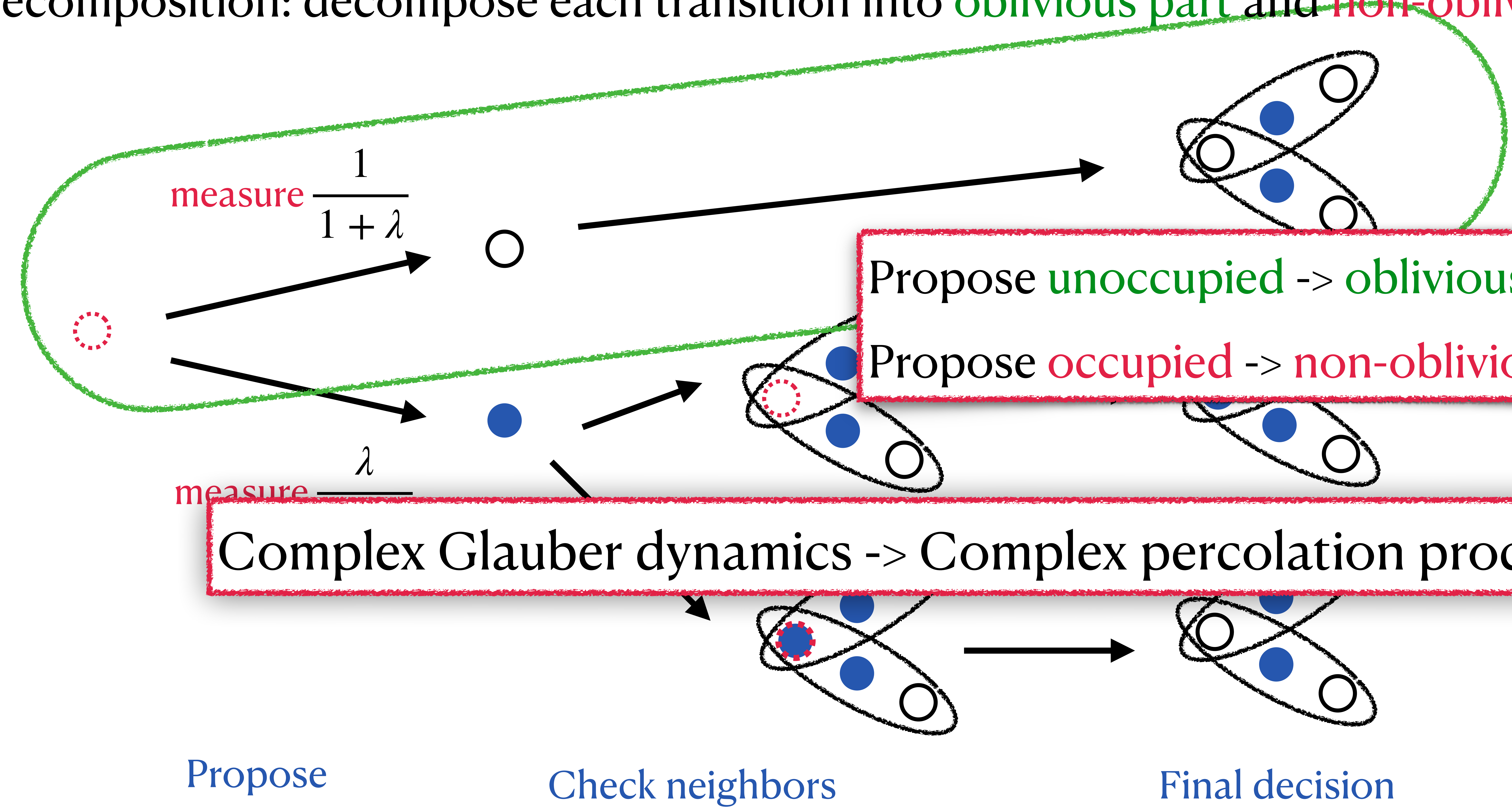

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Complex Glauber dynamics -> Complex percolation process

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Complex Glauber dynamics -> Complex percolation process

Zero-freeness

Complex convergence

Bounding the norms of complex measures for particular percolation clusters

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Complex Glauber dynamics -> Complex percolation process

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Complex convergence

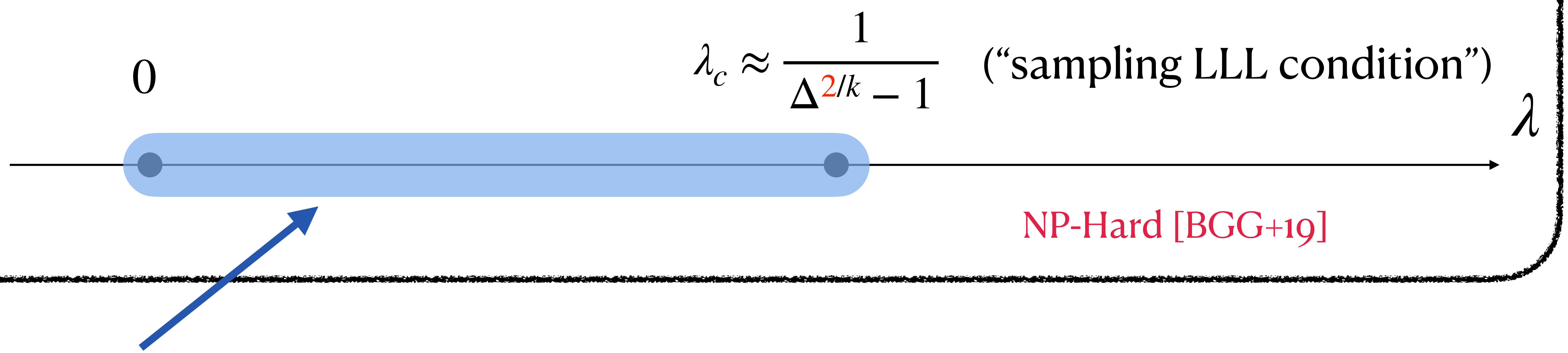
Bounding the norms of complex measures for particular percolation clusters

We use our zero-one law to bound these norms.

Technical contribution - complex percolation

We use complex percolation to analyze the complex systematic scan Glauber dynamics.

For k-uniform hypergraph with maximum degree Δ:

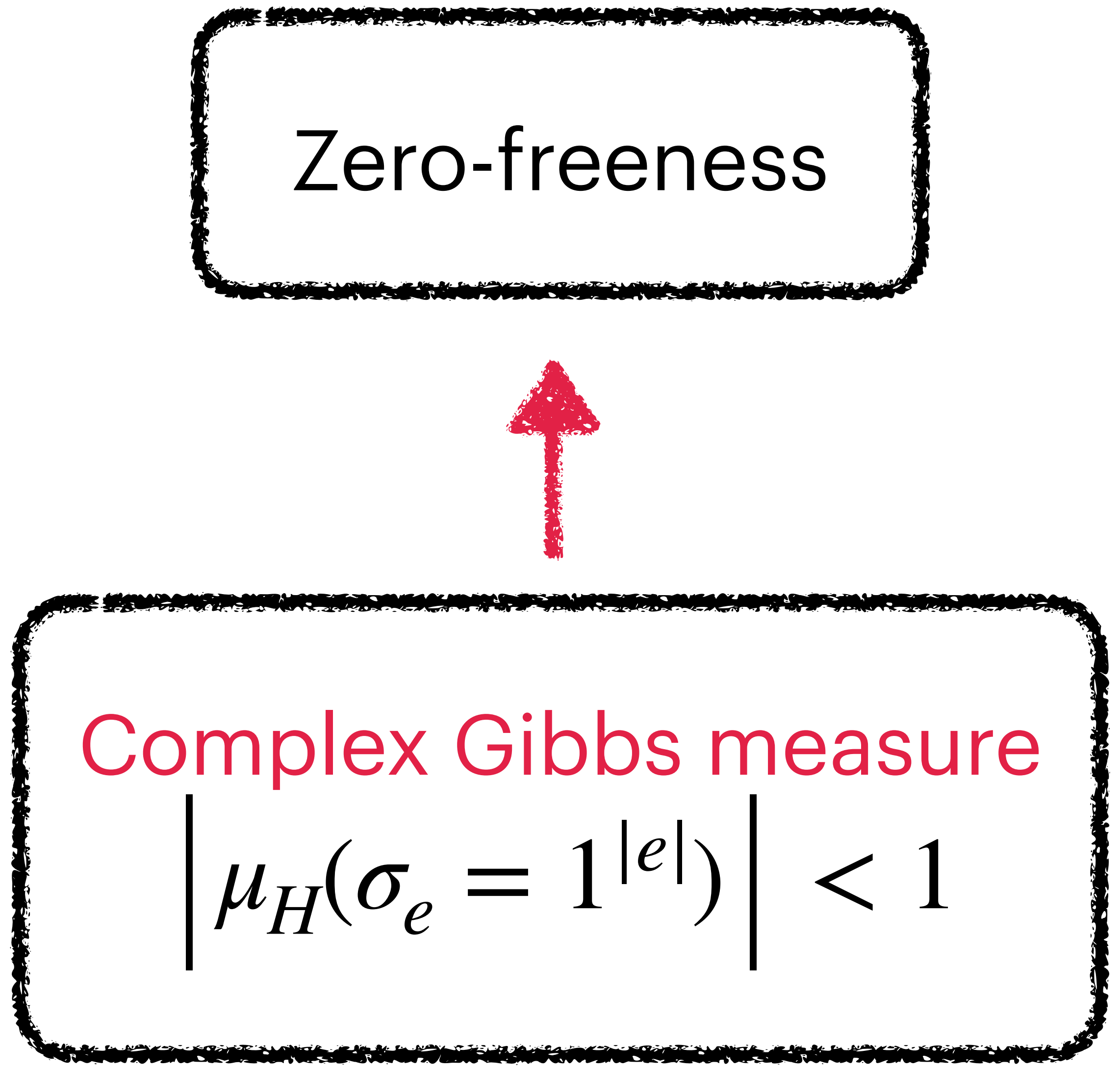


$\lambda_c \approx \frac{1}{\Delta^{2/k} - 1}$ ("sampling LLL condition")

NP-Hard [BGG+19]

We show in this strip, complex systematic scan Glauber dynamics converges and $Z(\lambda) \neq 0$.

Proof overview



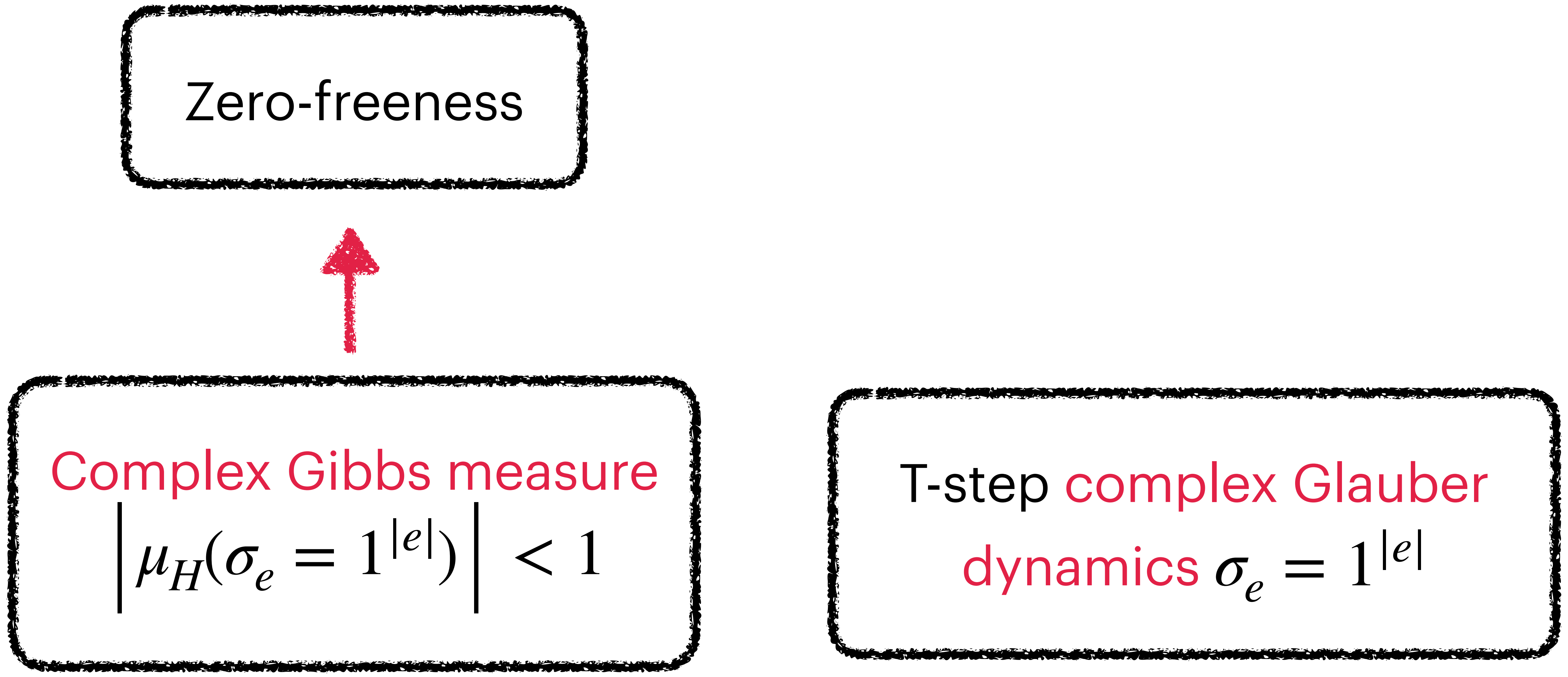
Zero-freeness

Complex Gibbs measure

$$\left|\mu_H(\sigma_e = 1^{|e|})\right| < 1$$

By standard edge-wise self-reducibility, it suffices to bound the norm of a complex marginal measure.

Proof overview



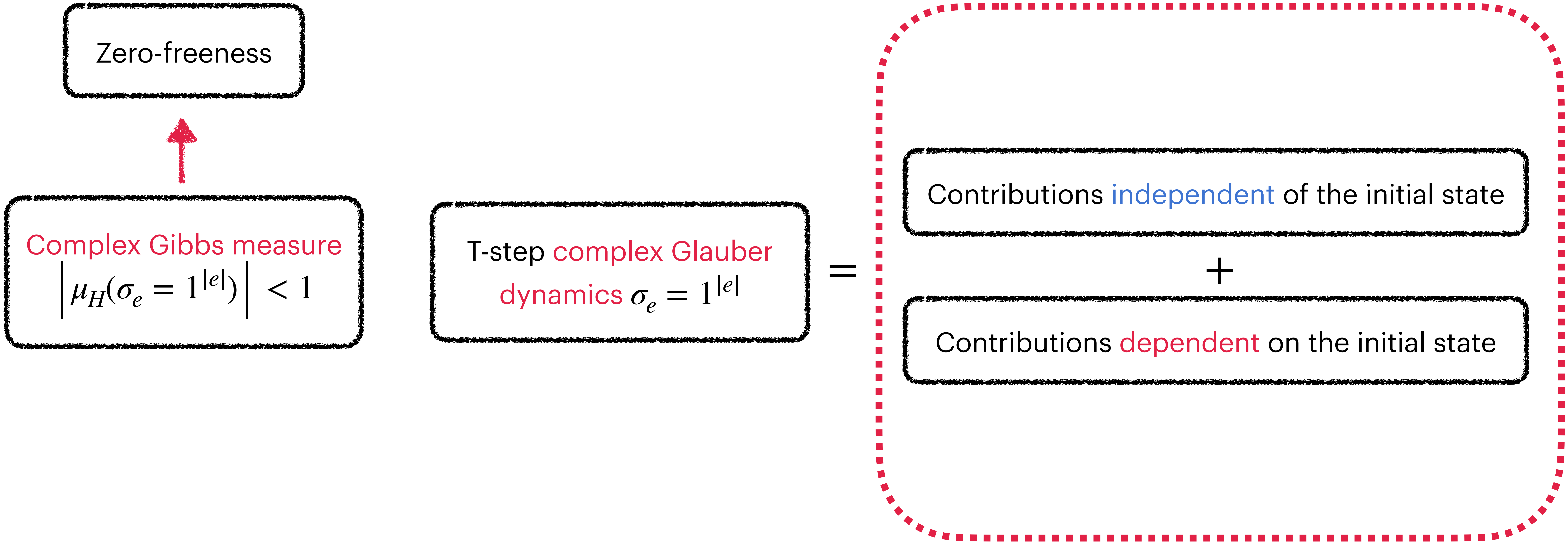
Zero-freeness

Complex Gibbs measure $\left|\mu_H(\sigma_e = 1^{|e|})\right| < 1$

T-step complex Glauber dynamics $\sigma_e = 1^{|e|}$

Expressing the complex Gibbs measure via the complex Glauber dynamics.

Proof overview



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T-step complex Glauber dynamics $\sigma_e = 1^{|e|}$

$=$

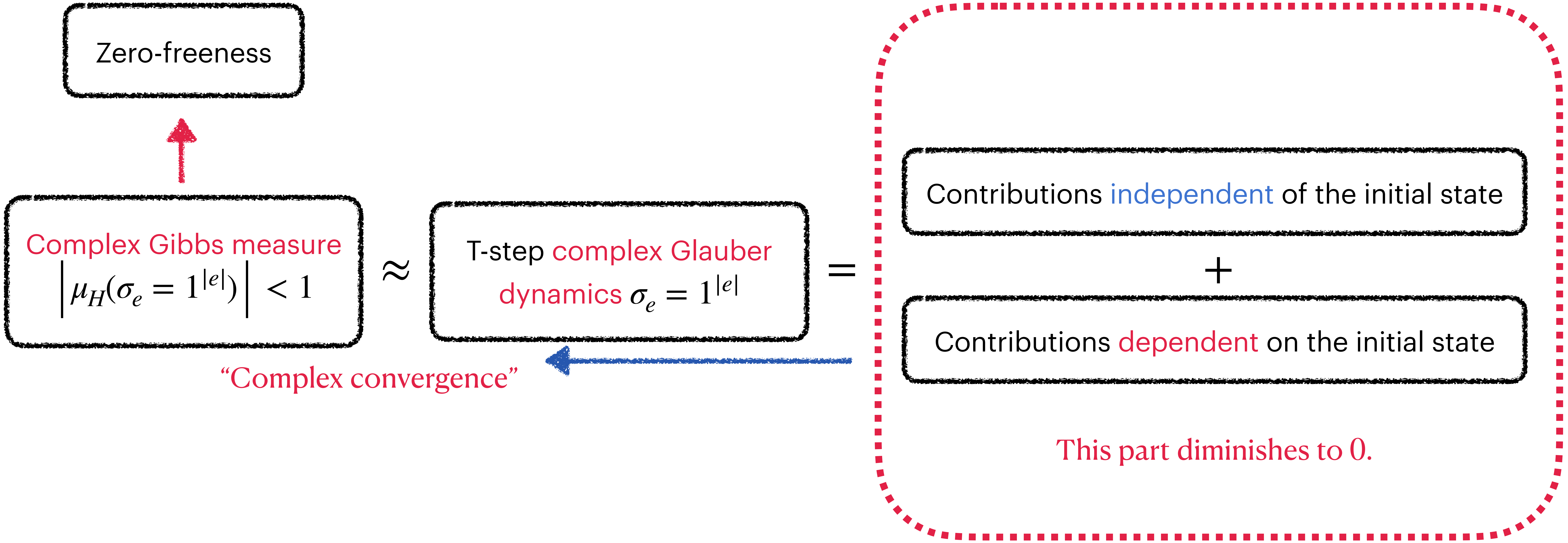
Contributions independent of the initial state

$+$

Contributions dependent on the initial state

Complex percolation

Proof overview



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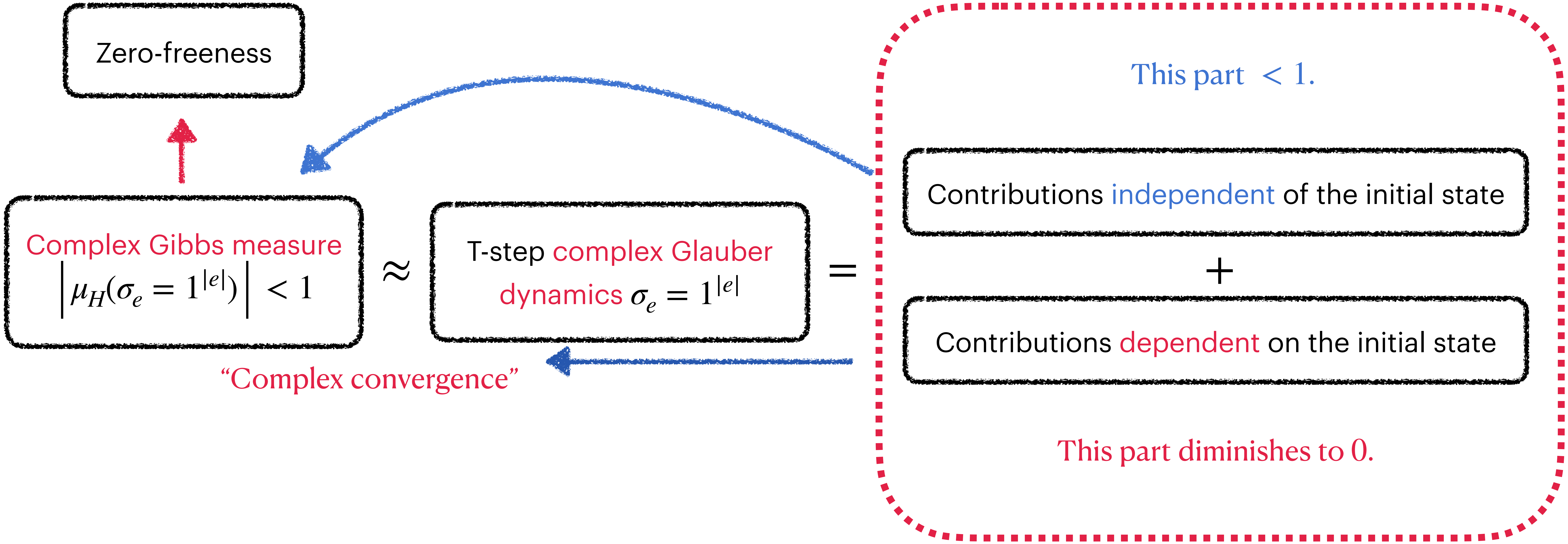
Contributions dependent on the initial state

This part diminishes to 0.

“Complex convergence”

Complex percolation

Proof overview



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“Complex convergence”

Complex percolation

Summary

We define the complex extensions of Markov chains and use it to improve the zero-free region of hardcore model on hypergraph.

As corollaries, we obtain efficient algorithms for:

1. approximating the partition function under the "sampling LLL condition",
2. approximating the number of t-size hypergraph independent sets.

Summary

We define the complex extensions of Markov chains and use it to improve the zero-free region of hardcore model on hypergraph.

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Open problems

1. Zero-freeness for general CSPs.
2. Does complex convergence imply zero-freeness?

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Thanks! Any questions?

Open problems

1. Zero-freeness for general CSPs.
2. Does complex convergence imply zero-freeness?